EXERCISES OF OPTICAL MEASUREMENTS

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EXERCISE 1

A CW laser radiation (λ=2.1 µm) is delivered to a Fabry-Pérot interferometer made of 2 identical plane and parallel mirrors at distance L=60 cm one from the other. One mirror is mounted on a PZT (PieZoelectric Transducer) with actuation factor K_{PZT}=40 nm/V.

a) Find the **values of reflectivity** of the mirrors to have transmission fringes with visibility values:

1) V_1=0.5
2) V_2=0.9
3) V_3=0.99

b) In the three cases considered, what are the values of **FSR** (**Free Spectral Range**) and **FWHM** (**Full Width Half Maximum**) of the Fabry-Pérot transmission profile?

c) Working with mirrors having reflectivity R=99.5 % and driving the PZT with a voltage ramp (0÷V_{MAX}) repeated every 100 µs, find the value of **V_{MAX} to scan 2 FSR**.

SOLUTION

We have to remember that, being $R=(R_1 \cdot R_2)^{1/2}$, the Airy transmission profile of a Fabry-Pérot interferometer as a function of the phase $\varphi=2\pi\nu/FSR$ is:

$$T(\nu) = \frac{(1-R)^2}{1+R^2-2R\cos(\varphi)}$$

$$T(\nu) = T_{MAX} = \frac{(1-R)^2}{1+R^2-2R} = 1 \quad for\ \cos(\varphi) = +1$$

$$T(\nu) = T_{min} = \frac{(1-R)^2}{(1+R)^2} \quad for\ \cos(\varphi) = -1$$

The visibility of transmission fringes is

$$V=\frac{T_{MAX}-T_{min}}{T_{MAX}+T_{min}}=\frac{1-\frac{(1-R)^2}{(1+R)^2}}{1+\frac{(1-R)^2}{(1+R)^2}}=\frac{(1+R)^2-(1-R)^2}{(1+R)^2+(1-R)^2}=\frac{4R}{2(1+R^2)}=\frac{2R}{1+R^2}$$

For $R\to 1$ we have $T_{min}\to 0$ and $V\to 1=100\%$.

1) $V=0{,}5\Rightarrow 2R=0.5\cdot(1+R^2)$
   $R^2-4R+1=0$ so $R_1=R=2\pm\sqrt{4-1}=2\pm\sqrt{3}$
   1st solution: $R_1=2+\sqrt{3}>1$, which is impossible.
   2nd solution: $R_1=2-\sqrt{3}\cong 0.26795\cong 26.8\%$

2) $V=0.9\Rightarrow 2R=0.9\cdot(1+R^2)$
   $R^2-2.222R+1=0$ so $R_2=R=1.111\pm\sqrt{1.2345-1}=1{,}111\pm\sqrt{0.2345}$
   1st solution: $R_2=1.111+\sqrt{0.2345}>1$, which is impossible.
   2nd solution: $R_2=1.111-\sqrt{0.2345}\cong 0.62679\cong 62.7\%$

3) $V=0.99\Rightarrow 2R=0.99\cdot(1+R^2)$
   $R^2-1.0101R+1=0$ so $R_3=R=1.0101\pm\sqrt{1.0203-1}=1{,}0101\pm\sqrt{0.0203}$
   1st solution: $R_3=1.0101+\sqrt{0.0203}>1$, which is impossible.
   2nd solution: $R_3=1.0101-\sqrt{0.0203}\cong 0.86761\cong 87.7\%$

a) The free spectral range is $FSR=c/2L=250$ MHz, not depending on the reflectivity values.
   The Finesse is

$$F=\frac{\pi\cdot\sqrt{R}}{1-R}$$

and, knowing F and FSR, the transmission line FWHM can be obtained as

$$\Delta\nu_{FWHM}=\frac{FSR}{F}$$

$R_1=26.8\%\to F_1\cong 2.2\to\Delta\nu_{FWHM,1}=112.5\ MHz$
$R_2=62.7\%\to F_2\cong 6.7\to\Delta\nu_{FWHM,2}=37.5\ MHz$
$R_3=87.7\%\to F_3\cong 22.1\to\Delta\nu_{FWHM,3}=11.3\ MHz$

b) In order to move/scan the transmission peaks of the Fabry-Pérot by one FSR, we have to change the length of the interferometer by $\Delta L_{FSR}=\lambda/2=1.05$ µm. This can be obtained driving the PZT with a voltage variation of $\Delta V_{FSR}=\Delta L_{FSR}/K_{PZT}\cong 25$ V.
   So, the voltage variation needed to scan two FSR is $V_{MAX}=2\cdot\Delta V_{FSR}\cong 50$ V.

   This value is independent of the values of reflectivity.

EXERCISE 2

A Fabry-Pérot resonator is composed of 2 equal mirrors, with reflectivity R=99.8 %, set apart at distance L=30 cm (in air or in vacuum).

a) Calculate the **Full Width Half Maximum** (**FWHM**) in transmission line.
b) Using a He-Ne laser source, evaluate the Fabry-Pérot **sensitivity** in terms of transmitted optical frequency variations ($\Delta\nu$) with respect to cavity length variations (ΔL).
c) How does this **sensibility change** if the source is a laser diode at 1550 nm?
d) We add a piezoelectric actuator, mounted on one mirror of the Fabry-Pérot, capable of changing the cavity length with an actuation coefficient K_{PZT}=1.8 nm/V. In the case of the laser diode at 1550 nm, which **voltage** we need to supply to the PZT to move the frequency of transmission by 5 MHz.

SOLUTION

a) The transmission of the Fabry-Pérot, as a function of the frequency, is given by Airy profile

$$T(\nu) = \frac{(1-R)^2}{1+R^2-2R\cos(\varphi)}$$

with $R=(R_1 \cdot R_2)^{1/2}$, or also $R=R_1=R_2$ when the mirrors have equal reflectivity $R_1=R_2$, and

$$\varphi = 2\pi \cdot \frac{L}{\frac{\lambda}{2}} = 2\pi \cdot \nu \cdot \frac{2L}{c} = 2\pi \cdot \frac{\nu}{\Delta\nu_{FSR}}$$

For high enough reflectivity values, the finesse (selectivity of the optical filter) is

$$F = \frac{\pi \cdot \sqrt{R}}{1-R} = \frac{\Delta\nu_{FSR}}{\Delta\nu_c} = 1569.225 \cong 1570$$

where $\Delta\nu_c$ is the Full Width Half Maximum and $\Delta\nu_{FSR} = \frac{c}{2L} = 500\ MHz$ is the Free Spectral Range of the resonator. So,

$$\Delta\nu_c = \frac{\Delta\nu_{FSR}}{F} = 318.6\ kHz \cong 319\ kHz \sim 320\ kHz$$

b) The resonance frequencies of the Fabry-Pérot are

$$\Delta\nu_m = m \cdot \frac{c}{2L}$$

where m is an integer number.

So,

$$\Delta\nu = -m \cdot \frac{c}{2L^2}\Delta L$$

and

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta L}{L}$$

Being $\nu = \frac{c}{\lambda} = \frac{3 \cdot 10^8 \frac{m}{s}}{633 \cdot 10^{-9}\ m} \cong 474\ THz$ and $L = 0.3\ m$ we obtain a sensitivity

$$S_{633\ nm} = \frac{\Delta\nu}{\Delta L} = -\frac{\nu}{L} = 1.58 \cdot 10^{15}\frac{Hz}{m} \cong 1.6\frac{MHz}{nm}$$

c) If the wavelength changes from 633 nm to 1550 nm (about a factor 2.4), the sensitivity changes consequently:

$$S_{1550\ nm} = \frac{\Delta\nu}{\Delta L} = -\frac{\nu}{L} = 6.45 \cdot 10^{14}\frac{Hz}{m} \cong 0.6\frac{MHz}{nm}$$

with $\nu = \frac{c}{\lambda} \cong 193\ THz$.

Besides, it must be

$$S_{1550\ nm} = S_{633\ nm} \cdot \frac{\nu_{1550\ nm}}{\nu_{633\ nm}} = S_{633\ nm} \cdot \left(\frac{633}{1550}\right) \cong S_{633\ nm} \cdot \left(\frac{1}{3}\right)$$

d) To move the frequency of transmission of 5 MHz, we have to produce a variation in wavelength

$$\Delta L = \frac{\Delta\nu}{S_{1550\ nm}} = \frac{5\ MHz}{0.6\frac{MHz}{nm}} = 8.33\ nm$$

which requires a variation in voltage

$$\Delta V = \frac{\Delta L}{K_{PZT}} = \frac{8.33\ nm}{1.8\frac{nm}{V}} = 4.63\ V$$

EXERCISE 3

A Fabry-Pérot interferometer mounts multi-dielectric mirrors with a power reflectivity R=99.5 % at the wavelength λ=1.55 μm. The first mirror is plane while the second is spherical with a radius of curvature *ROC*=50 cm. The two mirrors are mounted on a sledge, which allows regulating the inter-mirror distance from 2 cm up to 2 m. Moreover the plane mirror can be finely moved with a piezoelectric actuator with an actuation coefficient K_{PZT}=0.1 μm/V.

Calculate specific maximum and minimum values for these Fabry-Pérot parameters:

a) **Free Spectral Range** (***FSR***)
b) **Finesse** (***F***)
c) **Transmission linewidth** ($\Delta\nu_c$)
d) Possible values of the **diameter** (D_0) of the resonant mode on the plane mirror, at $1/e^2$ in terms of optical power with respect to peak power.
e) Working with similar mirrors and geometries, but varying the reflectivity *R*, which minimum value of reflectivity R_H shall we provide to have a transmission curve with FWHM $\Delta\nu_c$<0.1 kHz?
    - How much is the **Finesse** in this case?
    - How much is the value **Q** of the optical resonator?
    - Which are the **physical characteristics and non-idealities** of the mirrors that can represent a limit in the achievement of a linewidth so narrow (very high Finesse)?
f) For a length of the Fabry-Pérot of about 30 cm, can we have a **unitary transmission** of a laser at 1555.55 nm (resonance with a peak of transmission)?
    - What is the value of the integer number ***m*** which indicates the order of the longitudinal mode of the Fabry-Pérot excited by the wavelength of the laser?
    - If the wavelength of the laser shifts of 40 pm, which is the **variation in voltage** we have to apply to the piezoelectric to keep in resonance with the laser the same peak of transmission?

SOLUTION

a) The Free Spectral Range is

$$FSR = \frac{c}{2L}$$

Considering the two given lengths we obtain

$$FSR_{min} = \frac{c}{2L_{MAX}} = 75\ MHz$$
$$FSR_{MAX} = \frac{c}{2L_{min}} = 7.5\ GHz$$

Considering for L_{max} the limit imposed by the condition of stability in the plane-spherical resonator ($L < ROC$ e quindi $\bar{L}_{MAX} = ROC = 0.5\ m$) we obtain
$(\overline{FSR})_{min} = c/(2\bar{L}_{MAX}) = 300\ MHz$

b) The Finesse is

$$F = \frac{\pi\sqrt{R}}{1-R} \cong 626.7 \sim 626 \approx 630$$

The Finesse depends only on the reflectivity of the mirrors and <u>not</u> on the length of the resonator.

c) Starting from the expressions of Finesse $\left(F = \frac{\pi\sqrt{R}}{1-R}\right)$ and FSR $\left(FSR = \frac{c}{2L}\right)$, we obtain the linewidths

$$\Delta\nu_{c,MAX} = \frac{FSR_{MAX}}{F} \cong 12\ MHz$$
$$\Delta\nu_{c,min} = \frac{FSR_{min}}{F} \cong 120\ kHz$$

We can also compute

$$\Delta\bar{\nu}_{c,min} = \frac{\overline{FSR}_{min}}{F} \cong 480\ kHz$$

for $\bar{L}_{MAX} = ROC = 0.5\ m$, which is “more correct”.

d) The diameter we are looking for is $D = 2w_0$ on the plane mirror.

$$w_0 = \sqrt{\frac{\lambda L}{\pi}}\left[\frac{ROC}{L} - 1\right]^{\frac{1}{4}}$$

from $ROC = r(L) = L\left[1 + \left(\frac{\pi w_0^2}{\lambda L}\right)^2\right]$ ($r(L)$ is the radius of curvature of the Gaussian beam).

We have $w_{0,min} \cong 0$ near the region of instability of the resonator and so for $L \cong ROC = 0.5\ m$.
We have instead

$$w_{0,max} = w_0(L_{min}) = \sqrt{\frac{\lambda L_{min}}{\pi}}\left[\frac{ROC}{L_{min}} - 1\right]^{\frac{1}{4}}\Bigg|_{L_{min}=2\ cm} \cong 220\ \mu m$$

So,

$$D_{min} = 2w_{0,min} \cong 0$$
$$D_{MAX} = 2w_{0,MAX} \cong 440\ \mu m \approx 0.5\ mm$$

e) $\Delta\nu_C = \frac{FSR_{min}}{F} = \frac{FSR_{min}}{F(R_H)} \leq 0.1\ kHz$, working at the maximum distance between the mirrors.

- $F(R_H) \geq \frac{75\ MHz}{100\ Hz} = 750\ 000$ (both the mirrors can be plane and $L_{MAX} = 2\ m$)
- $F(R_H) \geq \frac{300\ MHz}{100\ Hz} = 3\ 000\ 000$ (considering a mirror with a $ROC = 0.5\ m = L^*_{MAX}$).

Both the computed values of Finesse are very high (and so they would be difficult to obtain in practical applications). We compute the reflectivity R_H of the mirror in case both the mirrors are plane, which it leads to a value of Finesse at least smaller.

We have to solve the equation

$$\pi \cdot R_H^{\frac{1}{2}} = 750\ 000 \cdot (1 - R_H) \xrightarrow{x = R_H^{\frac{1}{2}}} \pi x = 750\ 000(1 - x^2)$$

$$x^2 + \left(\frac{\pi}{750\ 000}\right)x - 1 = 0 \Rightarrow x = 0.999\ 998 \Rightarrow R_H = x^2 = 0.999\ 996 = 99.9996\%$$

In the considered case of $\Delta\nu_C \sim 100\ Hz$, being the factor of merit

$$Q = \frac{\nu}{\Delta\nu_c}$$

with $\nu = \frac{c}{\lambda} \cong 194\ THz$, it results $Q \cong 2 \cdot 10^{12}$ which is an extremely high value.

The absorption losses and the scattering from the mirrors, in the multidielectric layers which achieve the high reflectivity can limit the final value of reflectivity R in power. The transmission of the mirror is

$$T = 1 - R - A$$

It must be $T > 0 \Rightarrow R < 1 - A < 1 - A$ . .If the power losses are $A = 10^{-5}$, the reflectivity is limited to $R < 1 - A = 99.999\%$ and the finesse is limited to $F < 300\ 000$. It is almost impossible to achieve a Finesse equal to 3 000 000 and it is very difficult to obtain a Finesse of 750 000.

f) From the relation

$$\nu = m \cdot FSR$$

and being $\nu = \frac{c}{\lambda} \cong 192.858\ THz$ and $FSR = \frac{c}{2L} = 0.5\ GHz$, we can compute the integer number

$$m = \frac{\nu}{FSR} = 385716$$

From the relation

$$\frac{\Delta\nu}{\nu} = -\left(\frac{\Delta\lambda}{\lambda}\right) = -\left(\frac{\Delta L}{L}\right)$$

we obtain

$$\Delta L = \frac{\Delta\lambda}{\lambda} \cdot L = \frac{40}{1\ 555\ 550} \cdot 0.3\ m = 7.714\ \mu m$$

and this corresponds to

$$\Delta V_{PZT} = \frac{\Delta L}{K_{PZT}} = 77.14\ V$$

EXERCISE 4

In the given figure, we have a Michelson interferometer. The source is a DFB laser which emits a power $P_0 = 10\ mW$ at a wavelength $\lambda_0 = 1550\ nm$. Thanks to the presence of an optical insulator, the linewidth of the laser DFB is equal to 300 kHz. The interferometric signal is detected by a photodiode with spectral sensitivity (or responsivity) $\sigma = 0.5\frac{\text{A}}{\text{W}}$. The target is composed of a loudspeaker which is excited at the frequency $f_a = 100\ Hz$ by a sine wave with peak-to-peak voltage of 10 V (the sensitivity, in movement, of the loudspeaker is $2.635\ \frac{\mu m}{V}$).

The interferometer is unbalanced: $L_m = 1\ m$ and $L_r = 0.5\ m$.

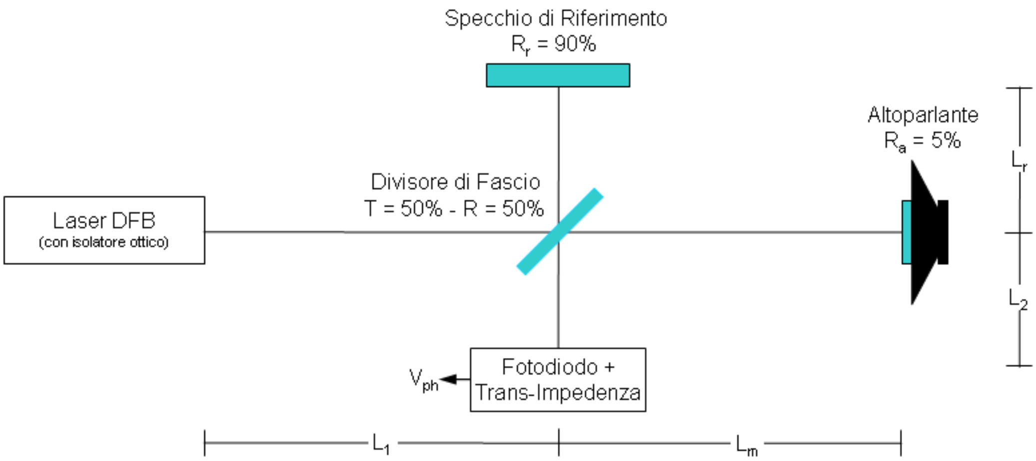
a) Compute the value of **maximum and minimum optical power P_{MAX} and P_{min}** which impinge on the photodiode.

b) The photocurrent generated by photodiode is amplified by a trans-impedance amplifier, with feedback resistance $R_r = 1\ k\Omega$. Derive the **expression of photovoltage V_f** in dependence of time, in particular $V_{f,MAX}$ and $V_{f,min}$.

c) Which is the **resolution** of the interferometer?
   - Compute the value of **phase shift** between the reference signal and measure for a complete excursion of the loudspeaker.
   - How many **interferometric fringes** this phase shift corresponds to?
   - Which is the **bandwidth B** which has to be guaranteed for a correct measurements of the vibration of the loudspeaker?

d) Due to a malfunctioning in the control of temperature of DFB laser, the wavelength of the device varies sinusoidally of 20 pm around the value λ_0 with frequency f_a. Will this variation cause an error in measurement of vibration of the loudspeaker? If so, how much is the **relative error** in the measured phase shift?

e) If we want to use the interferometer to measure the vibration with peak-to-peak amplitude from 1 nm to 100 nm, how do we have to **modify the readout scheme** of the interferometer? Does the resolution depend on L_m? On L_r?



SOLUTION

a) The reflected power of the loudspeaker is

$$P_l = \frac{P_0}{2} \cdot \frac{5}{100} = 0.25\ mW$$

and the reflected power by the mirror is

$$P_m = \frac{P_0}{2} \cdot \frac{90}{100} = 4.5\ mW$$

The optical power which recombine on the photodiode are

$$P_1 = \frac{P_l}{2} = 0.125\ mW$$

$$P_2 = \frac{P_m}{2} = 2.25\ mW$$

Knowing P_1 and P_2, the maximum and minimum powers of the beatings on the photodiode are

$$P_{MAX} = P_1 + P_2 + 2(P_1 P_2)^{\frac{1}{2}} \approx 3.435\ mW$$

$$P_{min} = P_1 + P_2 - 2(P_1 P_2)^{\frac{1}{2}} \approx 1.314\ mW$$

b) The photovoltage $V_f(t)$ is related to the optical power according to the following relation:

$$V_f(t) = R_r \cdot I_f(t) = R_r \cdot \sigma \cdot P(t)$$

The optical power P(t) varies in time because of the interference of the signals P_1 and P_2 with the following law:

$$P(t) = P_1 + P_2 + 2(P_1 P_2)^{\frac{1}{2}} \cdot \cos(2ks)$$

where $k = \frac{2\pi}{\lambda_0}$ is the wavenumber, $s = s_0 \cdot \cos(2\pi f_a t)$ is the temporal movement of the loudspeaker and $s_0 = 10 \cdot \frac{2.635}{2} \mu m \cong 13.2\ \mu m$.

Summarizing

$$V_f(t) = R_r \cdot \sigma \cdot \left\{P_1 + P_2 + 2(P_1 P_2)^{\frac{1}{2}} \cdot \cos\left[\frac{4\pi}{\lambda_0} \cdot s_0 \cdot \cos(2\pi f_a t)\right]\right\}$$

and so

$$V_{f,MAX} = R_r \cdot \sigma \cdot P_{MAX} = 1.7175\ V$$

$$V_{f,min} = R_r \cdot \sigma \cdot P_{min} = 658.5\ mV$$

c) Since it is a classic Michelson interferometer and there are no specifications about a particular method of elaboration of the signal $V_f(t)$, the resolution of the interferometer is

$$\frac{\lambda_0}{2} = 755\ nm$$

During a complete excursion of the loudspeaker, the totally collected phase shift between the reference signal and the measurement signal is

$$\phi_t = 2k \cdot 2s_0 = \frac{8\pi s_0}{\lambda_0} \cong 214\ rad$$

This phase shift corresponds to a number of interferometric fringes

$$N_t = \frac{\phi_t}{2\pi} = 34$$

We could also compute this value by dividing the complete movement, equal to s_0, by the movement an interferometric fringe generates in the signal $V_f(t)$, whici is equal to $\frac{\lambda_0}{2}$.

The measurement bandwidth B of the interferometer must guarantee a resolution of $2 \cdot 34 = 64$ interferometric fringes in a time

$$T_m = \frac{1}{f_a} = T_a = 10\ ms$$

So the bandwidth B must be

$$B \geq B_{min} = \frac{64}{T_m} = 6.4\ kHz$$

d) In an unbalanced interferometer, as the one we are considering, in correspondence of a variation in wavelength $\Delta\lambda = 20\ pm$ a variation/error of phase is generated according the following relation

$$\Delta\phi_\lambda = 2\pi \cdot [2(L_m - L_r)] \cdot \frac{\Delta\lambda}{\lambda_0^2} = 52.279\ rad$$

which corresponds to about 8 interferometric fringes.

The relative error on the measurement of the movement of the loudspeaker due to the malfunctioning of the temperature controller is

$$e_r = \frac{\Delta\Phi_\lambda}{\Phi_t} = 49\% \sim 50\%$$

e) When we want to measure movements/vibrations which are much less than $\frac{\lambda_0}{2}$, we have to modify the scheme we have analyzed: we have to keep the interferometer in quadrature (half fringe point) to exploit the linear segment of the characteristic of the transfer phase/movement – photovoltage $V_f(t)$.

In these conditions the resolution of the interferometer is given by the NED (Noise Equivalent Displacement). The contributions to the NED are the shot noise related to the detection of the optical signal, the finite linewidth of the laser and its interaction with the displacement of the interferometer.

The NED, which is related to the non-null linewidth of the source, is given by the relation

$$NED_p = (L_m - L_r) \cdot \left(\frac{\Delta\nu}{\nu_0}\right)$$

where $\nu_0 = \frac{c}{\lambda_0}$ and $q = e = 1.6 \cdot 10^{-19}\ C$.

The NED, which is related to the shot noise of detection, is given by the relation

$$NED_q = \frac{\lambda_0}{2\pi V} \cdot \sqrt{\frac{q \cdot B_{min}}{2 I_f}}$$

where $I_f = \sigma \cdot (P_1 + P_2)$.

Typically

$$NED_p < NED_q$$

so it is better to work with a balanced interferometer ($L_m = L_r$) so to cancel NED_p .

If we can balance the interferometer and work in half-fringe point, the resolution of the interferometer is

$$NED = NED_q = 3.73\ fm$$

EXERCISE 5

In figure 1 there is a Michelson interferometer. The source is a semiconductor laser with an emission wavelength $\lambda_0 = 800\ nm$, emitted power $P_{las} = 10\ mW$, linewidth $\delta\nu = 10\ MHz$. The beam splitter and the mirrors are ideal. The beam splitter has 50%-50% power reflection-transmission and the mirrors have 100% reflectivity. The photodiode has a spectral responsivity $\sigma = 0.5\frac{A}{W}$. The lengths of the arms are: $L_1 = 0.2\ m$, $L_3 = 1\ m$, $L_4 = 0.2\ m$. With mirror M_2 we measure the movement.

a) The temporal law according to which the length L_2 varies is given by the expression

$$L_2(t) = L_{2,0} + \Delta L_2(t)$$

where $L_{2,0} = 0.2\ m$ and the temporal dependence of $\Delta L_2(t)$ is shown in figure 2.

Draw, in a system of properly calibrated cartesian axes, the **temporal dependence of the photocurrent** $I_{ph}(t)$ which is generated by photodiode for $0 < t < 10\ ms$. How much is I_{ph} **for** $t = 4\ ms$ **and** $t = 7\ ms$.

b) We want to use the interferometer to measure small vibrations $\left(\ll \frac{\lambda}{2}\right)$ of the mirror M_2. For this purpose, we mount the mirror M_1 on a piezoelectric actuator and so we create a feedback system to keep the interferometer at half-fringe point.
Discuss the **NED (Noise Equivalent Displacement)** of this system, identifying the physical causes which generate the uncertainty on the physical quantity we want to measure.

c) Using again the interferometer as a vibrometer, determine which is the **minimum measurable movement** of M_2 (consider a bandwidth of 1 Hz) in these cases:
- $L_2 = 0.2\ m$ e $L_4 = 0.2\ m$
- $L_2 = 0.4\ m$
- $L_2 = 0.2\ m$ e $L_4 = 0.1\ m$

d) Suppose now that the emission wavelength of the laser source has the following dependence on the temperature

$$\lambda(T) = \lambda_0 + \xi \cdot T$$

where T is expressed in °C, $\lambda_0 = 800\ nm$ and $\xi = 50\frac{pm}{°C}$.

Supposing that the mirrors M_1 and M_2 are fixed, determine the **number of interferometric fringes** we observe on the photocurrent signal for a temperature variation $\Delta T = 10\ °C$ in the following cases:
- $L_2 = 0.2\ m$
- $L_2 = 0.6\ m$

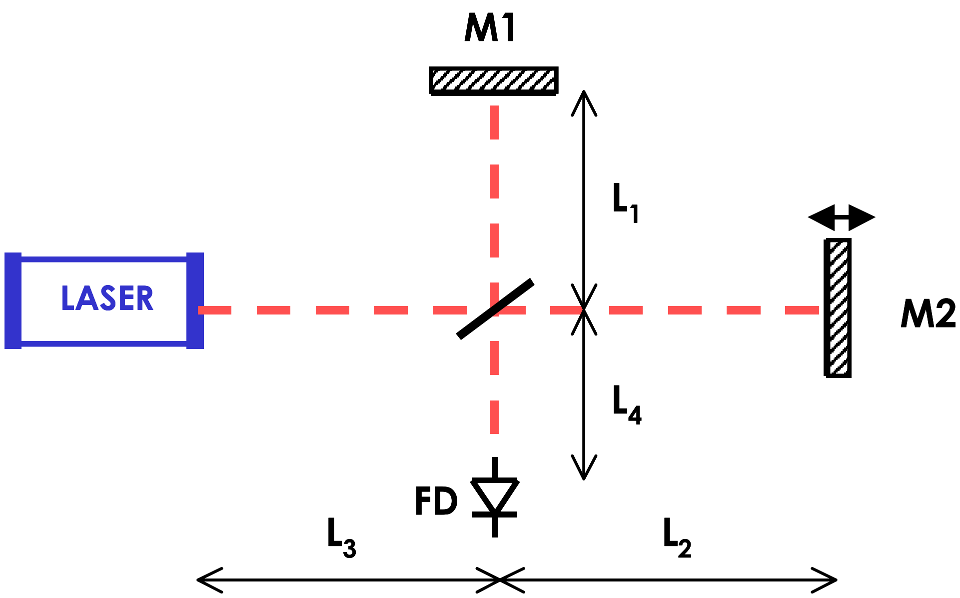


Figure 1

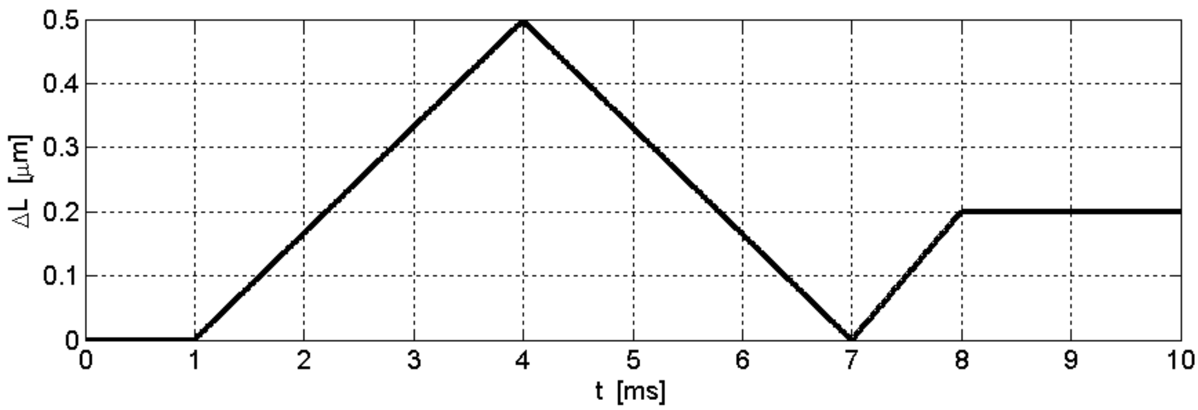


Figure 2

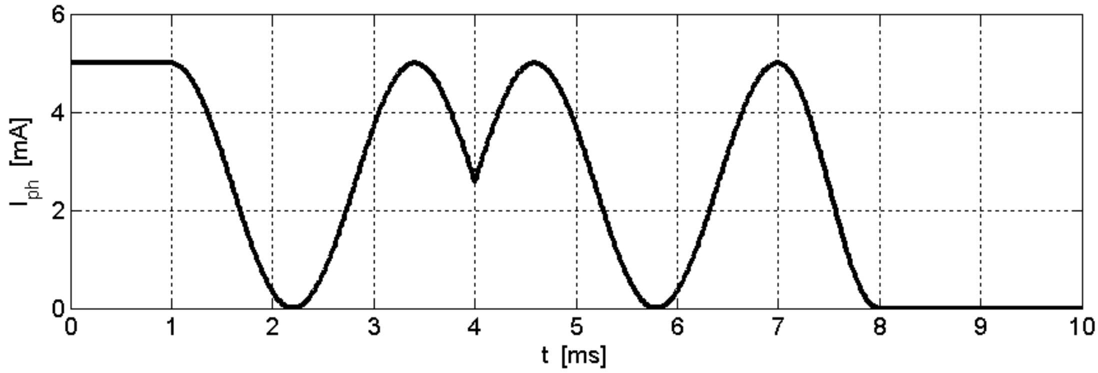
SOLUTION

a) The expression of the photocurrent is

$$I_{ph}(t) = \frac{I_0}{2} \cdot \left[1 + cos\left[2\left(\frac{2\pi}{\lambda}\right) \cdot \left(\Delta L_2(t)\right)\right]\right]$$

where $I_0 = \sigma \cdot P_{las} = 5\ mA$.

The graphic of $I_{ph}(t)$ is the following:



$$I_{ph}(t = 4\ ms) = \frac{I_0}{2} = 2.5\ mA$$
$$I_{ph}(t = 7\ ms) = I_0 = 5\ mA$$

b) We have to discuss the limit imposed by shot-noise associated to the photocurrent and the linewidth of the source (it influences the NED only in case $L_2 \neq L_1$). For all the details, look at the slides by Ing. Randone.

c)

$$NED = NED_p + NED_q$$
$$NED_p = (L_2 - L_1) \cdot \frac{\delta v}{v} = (L_2 - L_1) \cdot \delta v \cdot \frac{\lambda}{c}$$
$$NED_q = \left(\frac{\lambda}{2\pi}\right) \cdot \sqrt{\frac{qB}{I_0}} = \frac{0.8 \cdot 10^{-6}}{2\pi} \cdot \sqrt{1.6 \cdot \frac{10^{-19}}{5 \cdot 10^{-3}}} = 7.2 \cdot 10^{-16}\ m$$

where we have considered $B = 1\ Hz$.

For $L_2 = 0.2\ m$ and $L_4 = 0.2\ m$ we have $L_2 = L_1$ and so $NED_p = 0$, $NED = NED_q$

For $L_2 = 0.4\ m$ we have $NED \approx NED_p = (0.4 - 0.2) \cdot 10^7 \cdot 0.8 \cdot \frac{10^{-6}}{3 \cdot 10^8} = 5.33\ nm$

For $L_2 = 0.2\ m$ and $L_4 = 0.1\ m$ we have the same result as $L_2 = 0.2\ m$ and $L_4 = 0.2\ m$, because the length of L_4 is irrelevant for the NED_p.

d) When we have a (small) variation in the emission wavelength $\Delta\lambda = \xi \cdot \Delta T$, the interferometric signal can be written as

$I_{\downarrow}ph\ (t) = (I_{\downarrow}0)/2\ ([1 + cos[2"\ ("\ (-2((((/((_{\downarrow}0^{\uparrow}2\ ))"\ ("\ (L_{\downarrow}2\ I\ L_{\downarrow}1\ )\ ]\ ]$

because the phase variation is

$(\phi \;=\; 2((-2((((/((_{\iota}0^{\uparrow}2\;))\cdot(L_{\downarrow}2\;\mid\;L_{\downarrow}1)$

In case $L_2 = 0.2\ m$, we have $L_2 - L_1 = 0$ and

$\Delta\phi = 0$

So we do not observe interferometric fringes.

In case $L_2 = 0.6\ m$, we have

$$\Delta\phi = 2 \cdot \left(-2\pi \cdot 50 \cdot 10^{-12} \cdot \frac{10}{(0.8 \cdot 10^{-6})^2}\right) \cdot (0.6 - 0.2) = 3927\ rad$$

The number of interferometric fringes we observe is

$$\frac{\Delta\phi}{2\pi} = 625$$

EXERCISE 6

A Fabry-Pérot optical resonator is made of two equal mirrors, with reflectivity $R = 99{,}8\%$ at a distance $L = 30\ cm$ (in vacuum or in air).

a) Compute the **Full Widthat Half Maximum** (**FWHM**).

b) In case of He-Ne laser source, how much is the **sensitivity** in transmitted frequency (ν_0) with respect to the variations in the length of the cavity?

c) How much does the **sensitivity change** if the employed source is a laser diode at 1550 nm?

d) We mount a piezoelectric actuator on a mirror of the Fabry-Pérot, which is capable of varying the length of the cavity, with an actuator coefficient $K_{PZT} = 1{,}8 \frac{nm}{V}$. In case of the laser at 1550 nm, which **voltage** should we apply to the piezoelectric to shift the frequency of transmission of 5 MHz?

SOLUTION

a) The transmission of the Fabry-Pérot, depending on frequency, is given by the Airy pattern

$$T(\nu) = \frac{(1-R)^2}{1 + R^2 - 2R \cdot \cos\varphi}$$

$$R = R_1 = R_2$$

$$\varphi = \frac{2\pi \cdot L}{\frac{\lambda}{2}} = 2\pi \cdot \nu \cdot \frac{2L}{c} = 2\pi \cdot \frac{\nu}{\Delta\nu_{FSR}}$$

For sufficiently high values of reflectivity

$$F = \frac{\Delta\nu_{FSR}}{\Delta\nu_c} = \frac{\pi \cdot R^{\frac{1}{2}}}{1-R} = 1569.225 \cong 1570$$

where $\Delta\nu_c$ is the Full Width Half Maximum and $\Delta\nu_{FSR} = \frac{c}{2L} = 500\ MHz$ is the Free spectral Range of the resonator.

So, we can compute the linewidth as

$$\Delta\nu_c = \frac{\Delta\nu_{FSR}}{F} = 318.6\ kHz \cong 319\ kHz \sim 320\ kHz$$

Without using the definition of Finesse, the linewidth can be also computed also this way: the transmission peaks of the Fabry-Pérot are all equal and "analytically" for $\nu = 0$ there is the first peak with half width which can be computed when, for $\nu = \nu_{\frac{1}{2}}$, the Airy function reaches the value ½ for the first time. So

$$T(\nu) = \frac{(1-R)^2}{1+R^2-2R\cdot\cos\varphi} = 0.5$$

$$R\cdot\cos(\varphi) = 0.5\cdot(1+R^2)-(1-R)^2 = 2R-0.5-0.5R^2$$

$$\cos(\varphi) = 2-\frac{0.5}{R}-0.5R = 2-0.5\cdot\frac{1+R^2}{R} \cong 0.999998$$

$$\varphi = \arccos(\varphi) \cong 0.002\ rad = 2\pi\cdot\nu_{\frac{1}{2}}\cdot\frac{2L}{c}$$

$$\nu_{\frac{1}{2}} = \frac{c}{2L}\cdot\frac{\varphi}{2\pi} = FSR\cdot\frac{\varphi}{2\pi} = 500\ MHz\cdot\frac{2\cdot 10^{-3}}{6.28} = 159.3\ kHz$$

$$\Delta\nu_c = 2\nu_{\frac{1}{2}} = 318.6\ kHz \cong 319\ kHz \sim 320\ kHz$$

Another way to compute the linewidth $\Delta\nu_c$ is to express it as a function of the cavity lifetime

$$\tau_c = \frac{L}{c\gamma}$$

where γ represents the logarithmic losses of the resonator. By doing so we obtain

$$\Delta\nu_c = \frac{1}{2\pi\cdot\tau_c} = \frac{c\gamma}{2\pi\cdot L}\cdot\frac{\gamma_1+\gamma_2}{2} = \frac{c}{2\pi\cdot L}\cdot\frac{-\ln R_1-\ln R_2}{2} = \frac{c}{2\pi\cdot L}\cdot\frac{-\ln R-\ln R}{2} == -\frac{c}{2\pi\cdot L}\cdot\ln R$$

From it, we derive

$$\Delta\nu_c = \frac{-3\cdot 10^8 \frac{m}{s}}{2\pi\cdot 0.3\ m}\cdot \ln 0.998 \cong \frac{1}{6.28}\cdot(-10^{-9} Hz)\cdot(-2\cdot 10^{-3}) = 318.6\ kHz \cong\cong 319\ kHz \sim 320\ kHz$$

b) The resonance frequencies of the Fabry-Pérot are

$$\nu_m = m\cdot\frac{c}{2L}$$

where *m* is an integer number. So

$$\Delta\nu = -m\cdot\frac{c}{2L^2}\cdot\Delta L$$

or

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta L}{L}$$

Being $\nu = \frac{c}{\lambda} = \frac{3\cdot 10^8 \frac{m}{s}}{633\cdot 10^{-9}\ m} = 474\ THz$ and $L = 0.3\ m$ we obtain a sensitivity

$$S_{633\ nm} = \frac{\Delta\nu}{\Delta L} = -\frac{\nu}{L} = 1.58\cdot 10^{15}\frac{Hz}{m} \cong 1.6\frac{MHz}{nm}$$

c) If the wavelength passes from 633 nm to 1550 nm (about a factor 2.4), the sensitivity consequently scales

$$S_{1550\ nm} = \frac{\Delta\nu}{\Delta L} = -\frac{\nu}{L}\bigg|_{\nu=\frac{c}{\lambda}\cong 193\ THz} = 6.45\cdot 10^{14}\frac{Hz}{m} \cong 0.6\ \frac{MHz}{nm}$$

It has to be

$$S_{1550\ nm} = S_{633\ nm}\cdot\left(\frac{\nu_{1550\ nm}}{\nu_{633\ nm}}\right) = S_{633\ nm}\cdot\left(\frac{633}{1550}\right)\sim\left(\frac{1}{3}\right)S_{633\ nm}$$

d) To shift the transmission frequency of 5 MHz we have to produce the variation in length

$$\Delta L = \frac{\Delta\nu}{S_{1550\ nm}} = \frac{5\ MHz}{0.6\frac{MHz}{nm}} = 8.33\ nm$$

which requires a variation in voltage

$$\Delta V = \frac{\Delta L}{K_{PZT}} = \frac{8.33\ nm}{1.8\frac{nm}{V}} = 4.63\ V$$

EXERCISE 7

We have a LED source and an optical fiber. The LED source radiates with an angle θ with respect to the direction of the optical fiber ($I = I_0 \cdot \cos\theta$) and the numerical aperture of the optical fiber is NA_F .

Supposing that the maximum radiation of the LED is for $\theta = 0°$, compute the **efficiency** η , defined as the ratio between the power which is flowing in the optical fiber and the power which is emitted by the LED source.

SOLUTION

We have to distinguish 3 possible different situations:

a) The source is bigger than the optical fiber
b) The source is as big as the optical fiber
c) The source is smaller than the optical fiber

a) If the source is bigger than the fiber, only part of the emitted power can flow into the fiber, while the rest of the power is lost outside the optical fiber. Calling P_F the power flowing in the optical fiber and P_{LED} the emitted power of the LED,

$$\eta = \frac{P_F}{P_{LED}} = \frac{B_{LED} \cdot A_F \cdot \pi \cdot NA_F^2}{B_{LED} \cdot A_{LED} \cdot \pi} = \frac{A_F}{A_{LED}} \cdot NA_F^2$$

where B_{LED} is the brightness of the source, A_F is the area of the fiber and A_{LED} is the area of the LED source.

b) If the LED source has the same dimensions as the optical fiber, we have the maximum of the efficiency

$$\eta = \frac{P_F}{P_{LED}} = NA_F^2 = \eta_{MAX}$$

c) When we diminish the dimensions of the LED source, we diminish also the emitted power. So the efficiency remains constant to the value

$$\eta = \eta_{MAX}$$

EXERCISE 8

At a distance $z_o = 40\ cm$ from a diffusing target ($\delta = 1$) there is a He-Ne laser ($\lambda = 633\ nm$) emitting a beam with power $P_L = 1\ mW$ and beam waist $w_0 = 50\ \mu m$.
Compute:

a) the **diameter** D of the beam spot on the target
b) the **transversal and longitudinal dimensions of the speckles** at distance z_0
c) the **efficiency** η_{sp}, defined as the ratio between the speckle power and the laser power

Now we substitute the diffusing target with a corner cube.

d) Compute the **efficiency** η_{cc},defined as the ratio between the power of the corner cube and the laser power

SOLUTION

a) The divergence of the beam is

$$\theta_L = \frac{\lambda}{\pi \cdot w_0} = 4.03 \cdot 10^{-3}\ rad$$

The dimension of the spot is

$$D = 2 \cdot \theta_L \cdot z_0 = 3.22\ mm$$

b) The transversal dimension of the speckle can be easily computed as

$$s_{t_{@z_0}} = \lambda \cdot \frac{z_0}{D} = \lambda \cdot \frac{z_0}{2 \cdot \theta_L \cdot z_0} = \frac{\lambda \cdot z_0 \cdot \pi \cdot w_0}{2 \cdot \lambda \cdot z_0} = \frac{\pi}{2} \cdot w_0 = 78.54\ \mu m$$

while the longitudinal dimension of the speckle has value

$$s_{l_{@z_0}} = \lambda \cdot \left(\frac{2 \cdot z_0}{D}\right)^2 = 39\ mm$$

c) The brightness of the diffusing target is

$$B_D = \frac{P_L \cdot \delta}{\pi^2 \cdot \left(\frac{D}{2}\right)^2} = 39.09 \ \frac{W}{sr \cdot m^2}$$

The area of the speckle is

$$A_{sp} = \pi \cdot \left(\frac{s_t}{2}\right)^2 = 1.54 \cdot 10^{-9} \ m^2$$

and the solid angle is

$$\Omega_{sp} = \frac{\pi \cdot \frac{D^2}{4}}{z_0^2} = 5.09 \cdot 10^{-5} \ sr$$

The speckle power is

$$P_{sp} = B_D \cdot A_{sp} \cdot \Omega_{sp} = \frac{P_L \cdot \delta}{\pi^2 \cdot \left(\frac{D}{2}\right)^2} \cdot \pi \cdot \left(\frac{s_t}{2}\right)^2 \cdot \frac{\pi \cdot \frac{D^2}{4}}{z_0^2} = 9.64 \ pW$$

So the efficiency assumes the value

$$\eta_{sp} = \frac{P_{sp}}{P_L} \cong 9.64 \cdot 10^{-9} \sim 10^{-8}$$

d) The brightness of the corner cube is equal to the brightness of the laser

$$B_{cc} = B_L = \frac{P_L}{\pi^2 \cdot w_0^2 \cdot \theta_L^2}$$

So

$$\eta = \frac{P_{cc}}{P_L} = \frac{\pi \cdot w_0^2}{\pi \cdot \left(\frac{2D}{2}\right)^2} = 2.41 \cdot 10^{-4}$$

Exercise 9

We are given a directional coupler made of optical fiber ($n_F = 1.5$) 50%-50%. On one input port there is a laser diode emitting a power $P_L = 10\ mW$ at $\lambda_0 = 850\ nm$ and a lens, on the other input port there is a photodiode, while both the output ports are closed to ground. On one of the output arms of the directional coupler there is a piezoelectric coil composed of 20 rounds, with internal radius equal to $R_{ext} = 30\ mm$ and thickness equal to $t = 3\ mm$ for which we define $d = \frac{\delta R}{R_{ext}} \cdot \frac{1}{E} = 10^{-9} \frac{m}{V}$ and to which a voltage source with triangular shape (peak voltage $V_{pp} = 20\ V$ and frequency $f = 200\ Hz$).

Solution

First of all, we compute

$$\Delta L_F = M_F \cdot 2\pi(R(t) - R_{ext}) = M_F \cdot 2\pi \cdot \delta R$$

where

$$\delta R = d \cdot E(t) \cdot R_{ext} = d \cdot \frac{V_{pp}}{t} \cdot R_{ext}$$

The associated phase shift is

$$\Delta\phi = 2k \cdot \Delta L_F \cdot n_F$$

So the number of fringes

$$N_{fringes} = \frac{\Delta\phi}{2\pi} = \frac{\Delta L_F \cdot n_F}{\frac{\lambda_0}{2}} = 887$$

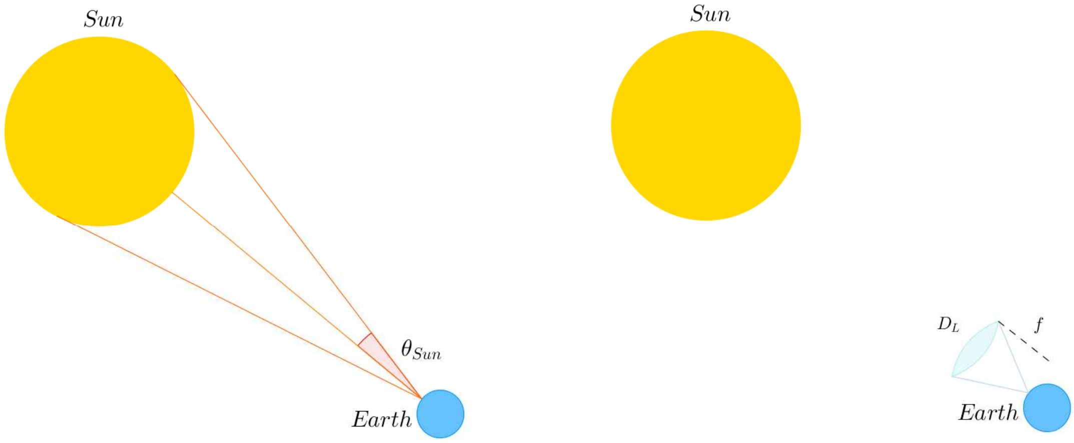
T_{fringe} is simply

$$T_{fringe} = \frac{T_{interferometric\ signal}}{N_{fringes}} = \frac{T}{2N_{fringes}}$$

EXERCISE 10

We consider two different situations (look at the figure). In the first one the sun light impinges directly the Earth. The half-angle between the Sun and the Earth is $\theta_{sun} = 4.5\ mrad$. In the second situation, at distance f (focal distance) from the Earth there is a convergent lens with diameter D_L .

Compute the **confinement factor** CF , defined as the ratio between the electric field in the second case E_2 and the electric field in the first case E_1, and **its maximum value**.



SOLUTION

The electric field in the first case is

$$E_1 = B_{sun} \cdot \pi \cdot \theta_{sun}^2$$

The electric field in the second case is

$$E_2 = B_{sun} \cdot \pi \cdot NA_L^2$$

where NA is the numerical aperture of the lens.

So the collimation factor is

$$CF = \frac{B_{sun} \cdot \pi \cdot NA_L^2}{B_{sun} \cdot \pi \cdot \theta_{sun}^2} = \frac{NA_L^2}{\theta_{sun}^2}$$

The maximum value of the confinement factor, having fixed θ_{sun} to a specific value, is obtained in correspondence of the maximum value of the numerical aperture ($NA_L = 1$)

$$CF_{MAX} = \frac{NA_L^2}{\theta_{sun}^2}\bigg|_{NA_L=1} = \frac{1}{4.5 \cdot 10^{-3}\ rad} \cong 50000$$