GRAPHIC REPRESENTATION OF EXPERIMENTAL RESULTS

INTERPOLATION AND FITTING CURVES

Sistemi di Acquisizione Dati
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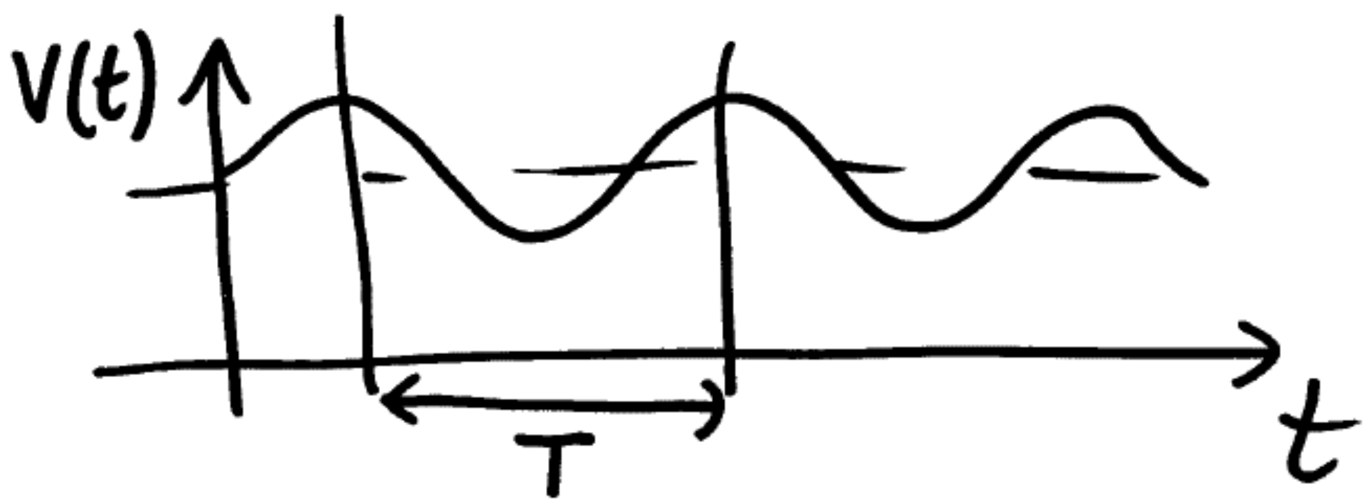
Graphic Representation

- "Overview" of a **magnitude** as a function of **time** or of another parameter
- Typically using **coordinate axes** that must report the description of the represented magnitude and if necessary also its **units of measurement**



Types of Graphs

- When on the axes, **numerical values** appear, you must always indicate the corresponding measurement units. The graph is said **QUANTITATIVE**
- Otherwise the diagram is **QUALITATIVE** and can serve to indicate the **trends** or **tendency**

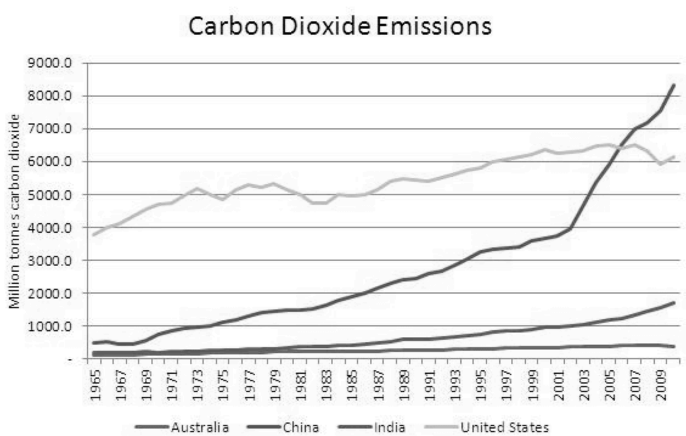


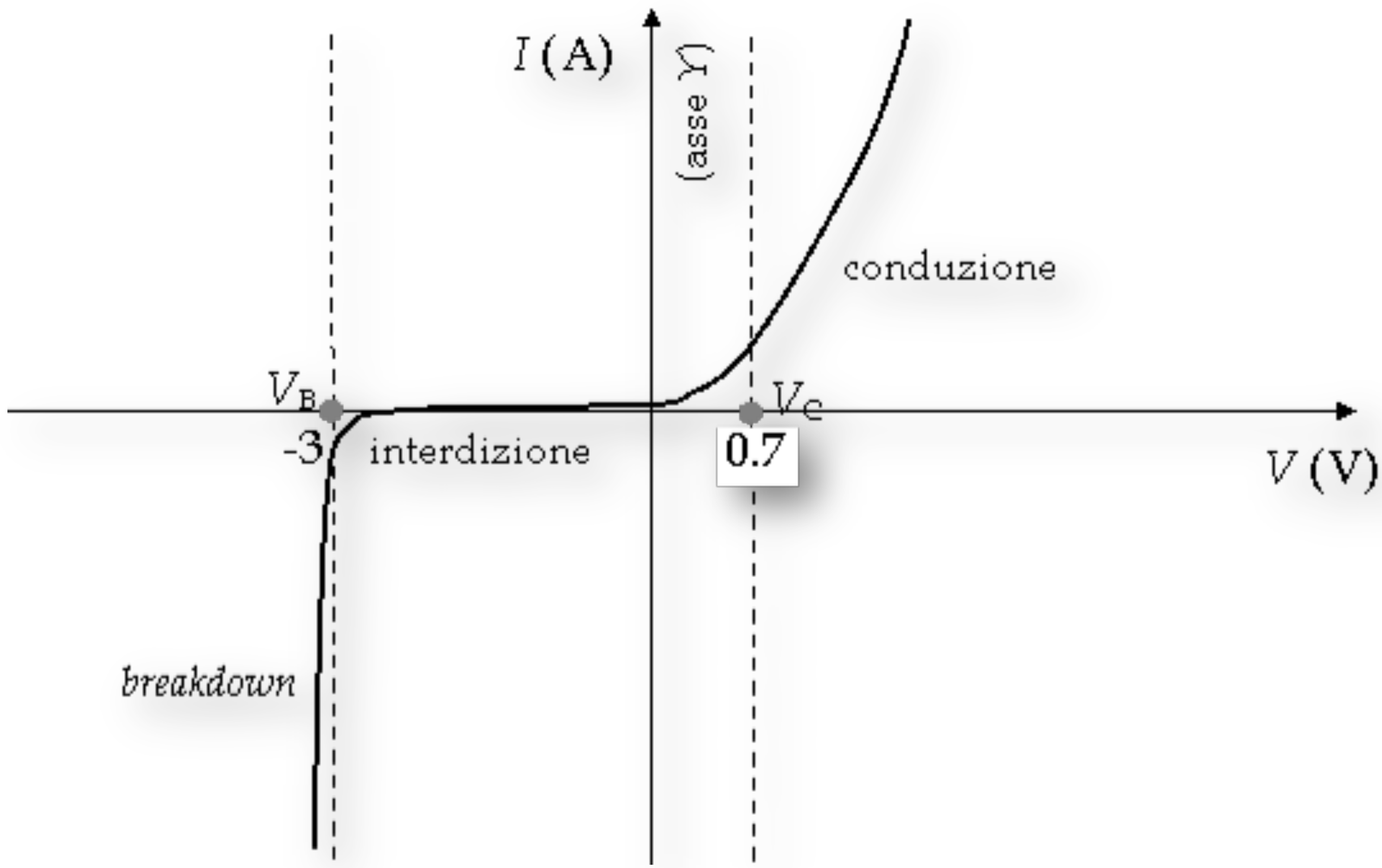
Chart in a CARTESIAN PLANE

- ABSCISSA (axis X): independent variable or command or input
- ORDINATE (axis Y): dependent variable or output variable

Typically $u(x_i) \ll u(y_i)$, so the command variable is known with good accuracy (negligible uncertainty) while the variable output presents greater uncertainty

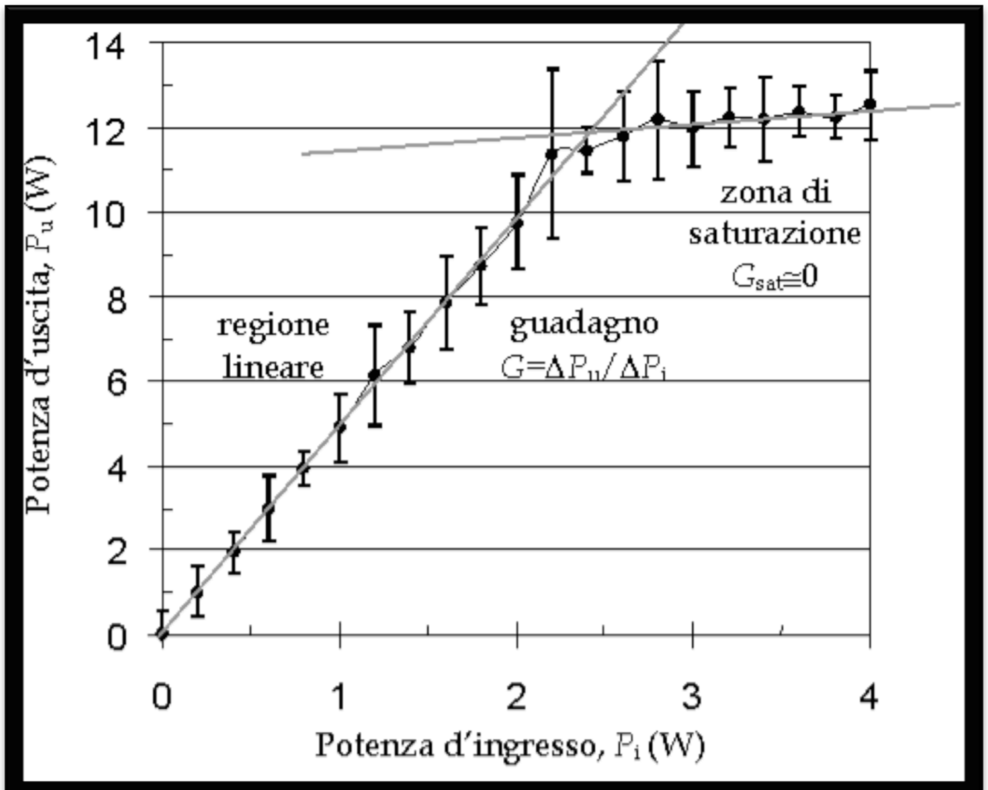
Many times the uncertainties of inputs and outputs are not specified, but along with the noise on the data translate into a "**scatter of the experimental points**"

Current-voltage characteristic for a Zener diode



Graphic representation of the dispersion (uncertainty): Error Bars

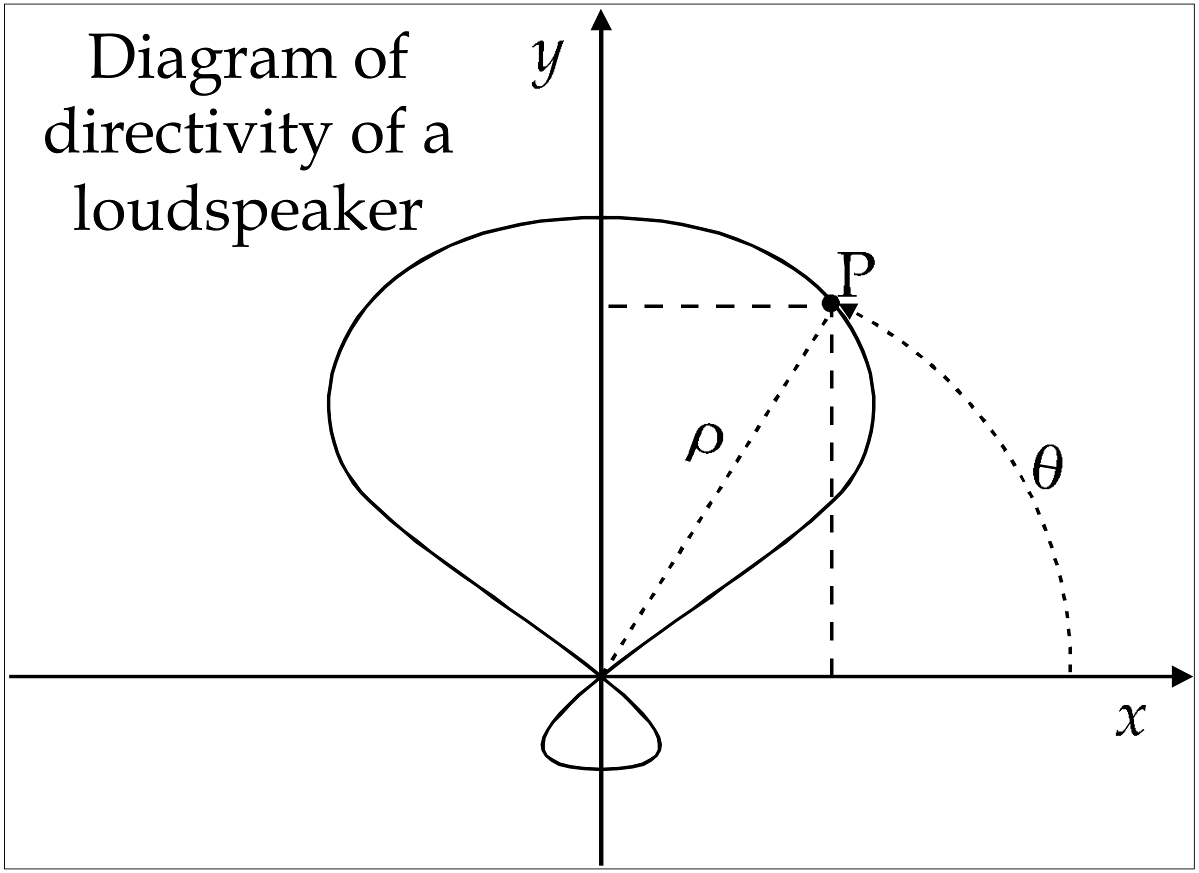
Input-output characteristic of an electronic amplifier.
Error bars indicate a confidence interval, which must be specified: for example ± 1σ (68%) or ± 2σ (95.5%)



Polar diagram

Radial coordinate: $\rho = (x^2 + y^2)^{1/2}$

Angolar coordinate: $\theta = \text{arctg}(y/x)$ per $x \geq 0$



$x = \rho \cos(\theta)$

$y = \rho \sin(\theta)$

$\rho(\theta)$ may also indicate the power radiated by an antenna

Logaritmic scale

Useful for display sizes that vary by several orders of magnitude, with constant relative detail:
equally spaced points on a logarithmic scale are in the same scale in a linear relationship.

$\boldsymbol{z|_{\log}=\log_B(z/z_0)}$ B is the base and z_0 is the reference

Very common dB e dBm (with B=10)

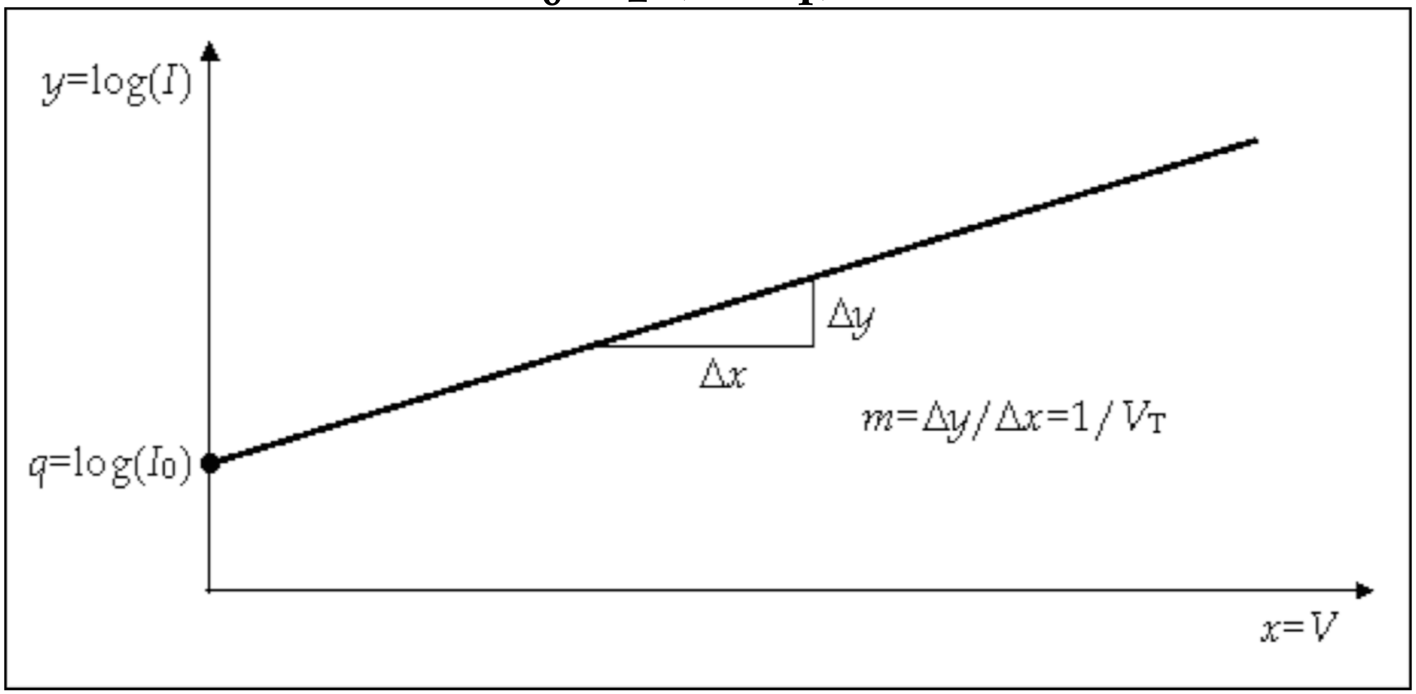
$P|_{dB}=10\log_{10}(P/P_0)$

$A|_{dB}=20\log_{10}(A/A_0)$

$P|_{dBm}=10\log_{10}[P/(P_m)]$ con P_m=1 mW

Semilogaritmic diagram (log-lin)

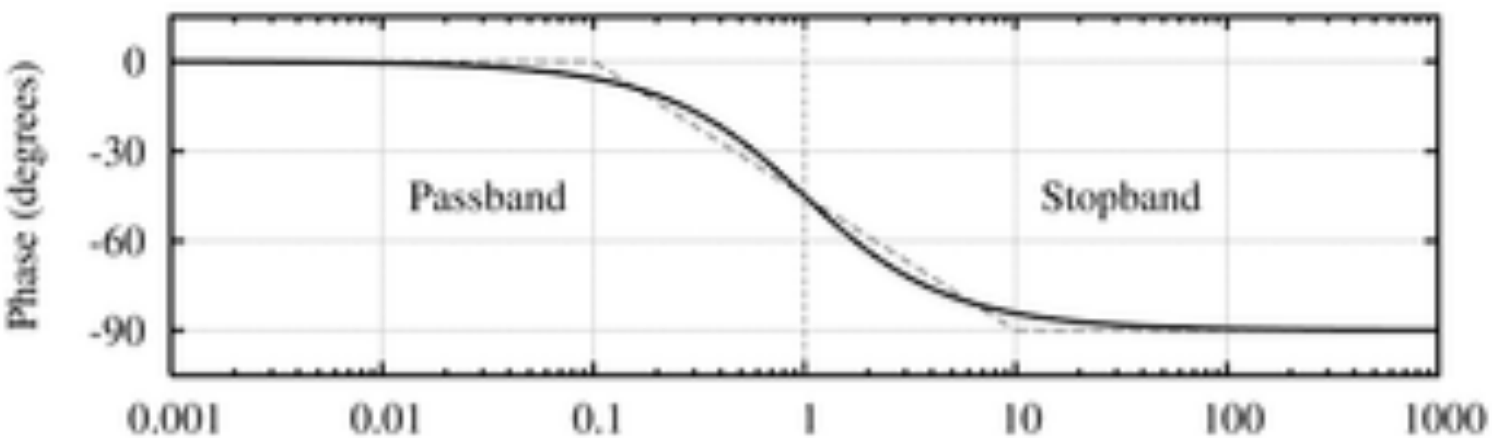
Diagram semilog-*y* for the *I*-*V* curve of a semiconductor diode in forward bias: $\boldsymbol{I=I_0\exp(V/V_T)}$



$$y = \log(I) = (1/V_T)\times V+\log(I_0) = mx+q$$

$$m = (1/V_T) \qquad q=\log(I_0)$$

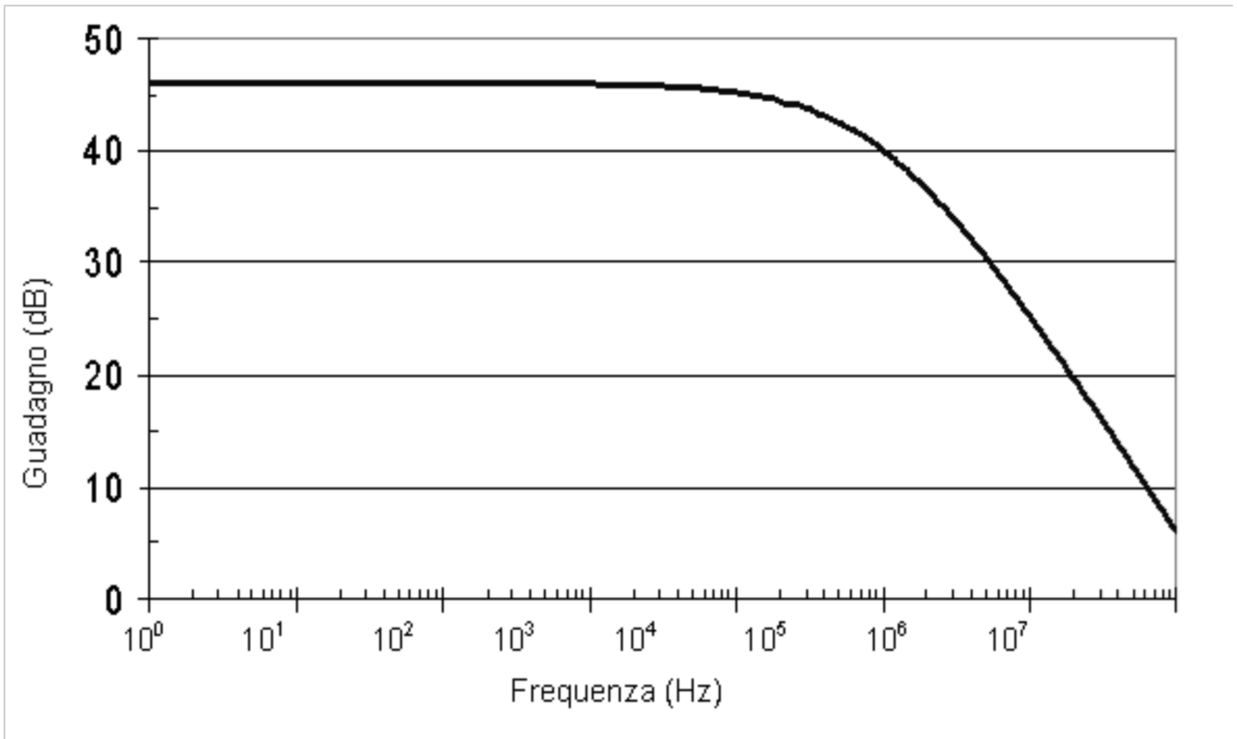
Semilogaritmic Diagram (lin-log): phase Bode diagram



6 decades (from 1 mHz up to 1 kHz)

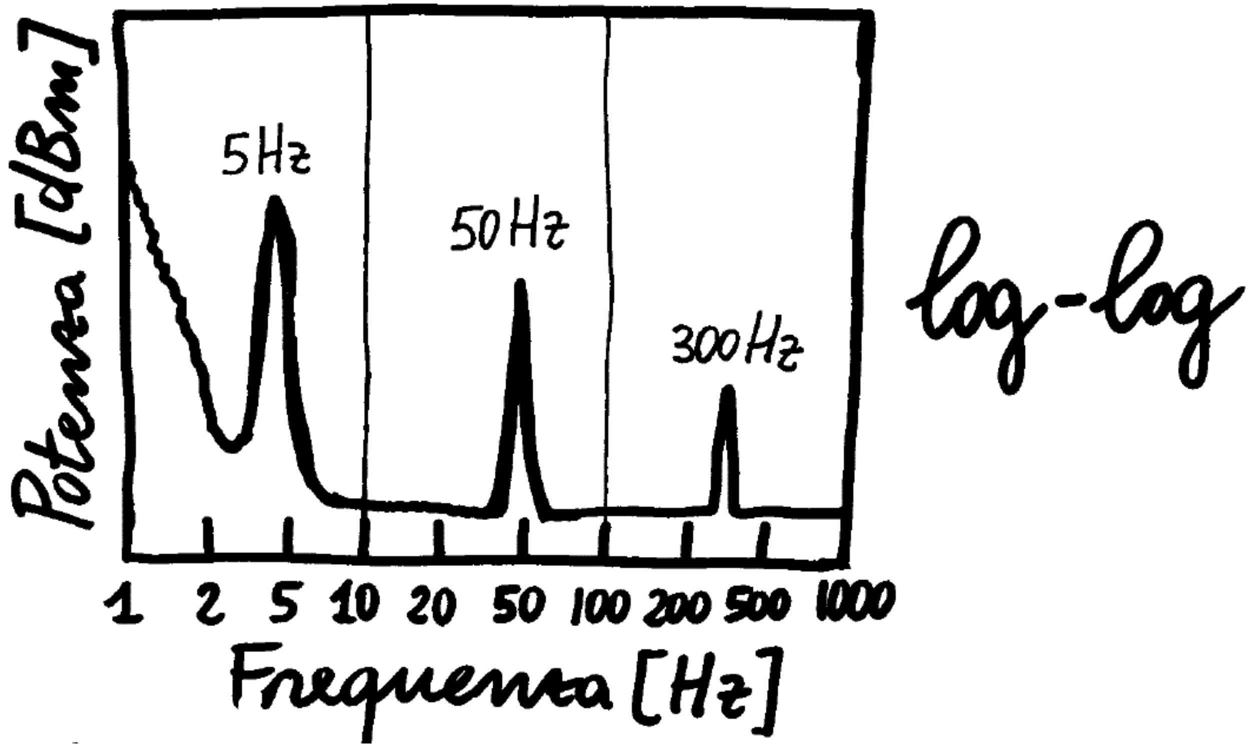
Phase shift in degrees or radians as a function of the frequency reported in logarithmic scale (wide dynamic).

Bilogaritmic Diagram (log-log): amplitude Bode diagram



Amplitude or gain in dB as a function of the frequency reported in logarithmic scale: you can find typical slope (e.g., -20 dB / dec).

Bilogaritmic Diagram (log-log): signal power spectrum



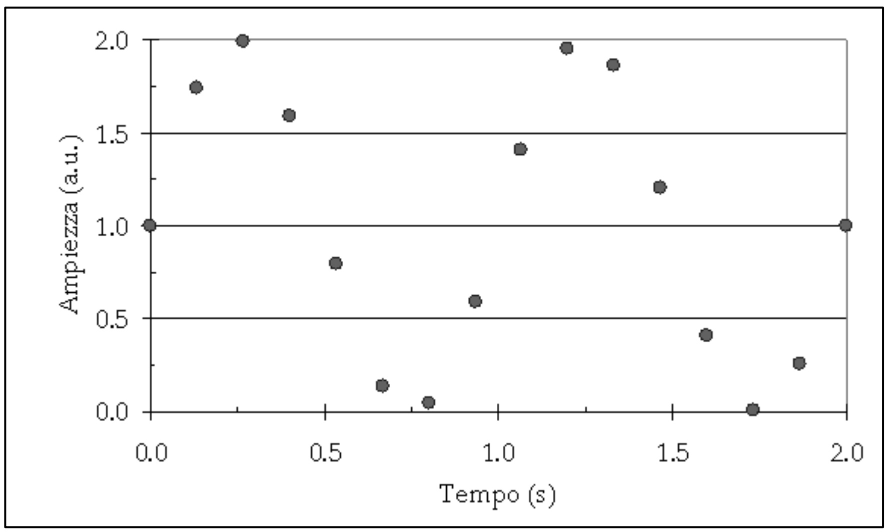
Wide dynamic of frequency and power can be displayed on the same diagram.

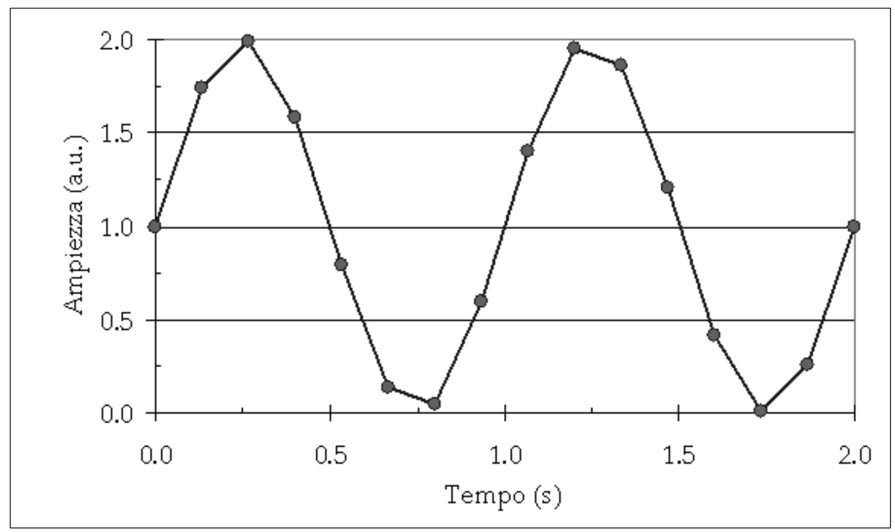
Interpolation

- Measurement: set finite and discrete of experimental values.
- These discrete experimental points are typically the values assumed by the object of measurement varying one or more control parameters (input quantity).Or these points are discrete samples acquired over time.
- The representation is more readble when we put a "filler" **or interpolation between two adjacent data experimental points.**
- Interpolating function: is a continuous function, which passes through the two points and provides us with the presumed trend (interpolated) of the input-output relationship.

Linear interpolation

It is the simplest possible interpolation: consists in joining the points with a broken line (set of segments of straight lines that pass through two adjacent points).

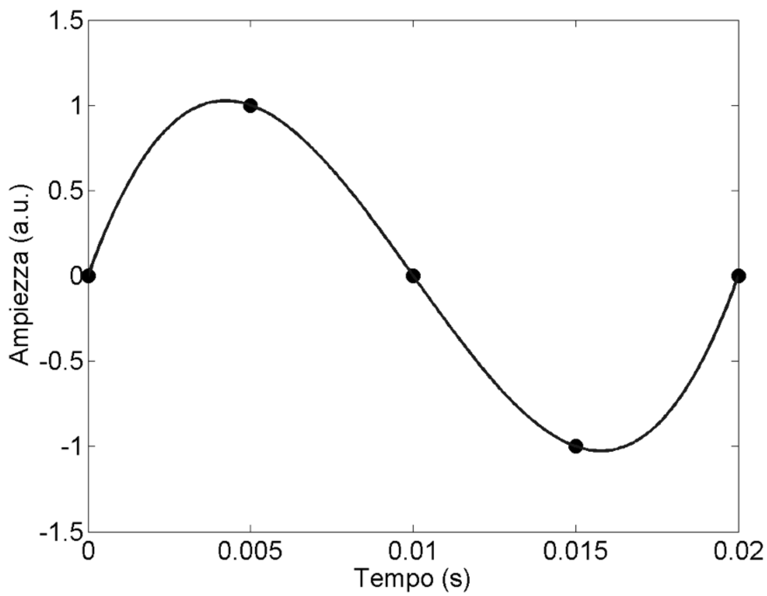


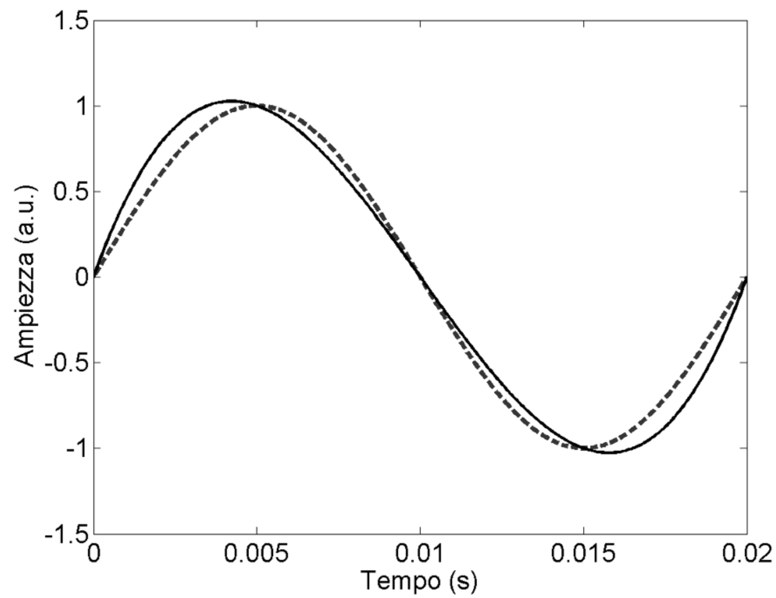


It can not achieve good signal reconstruction because it does not exploit the information of the preceding and following points.

Cubic polinomial interpolation

It is the curve passing through the experimental points, while maintaining the continuous first derivative and second.





It has the visual effect of a "smoothed line". It can be obtained with different boundary conditions (in the two extreme points of the interval available data).

Cubic polinomial interpolation

$$S_i(x) = ax^3 + bx^2 + cx + d$$

Represents the *spline* that interpolates the *S* function, the conditions are:

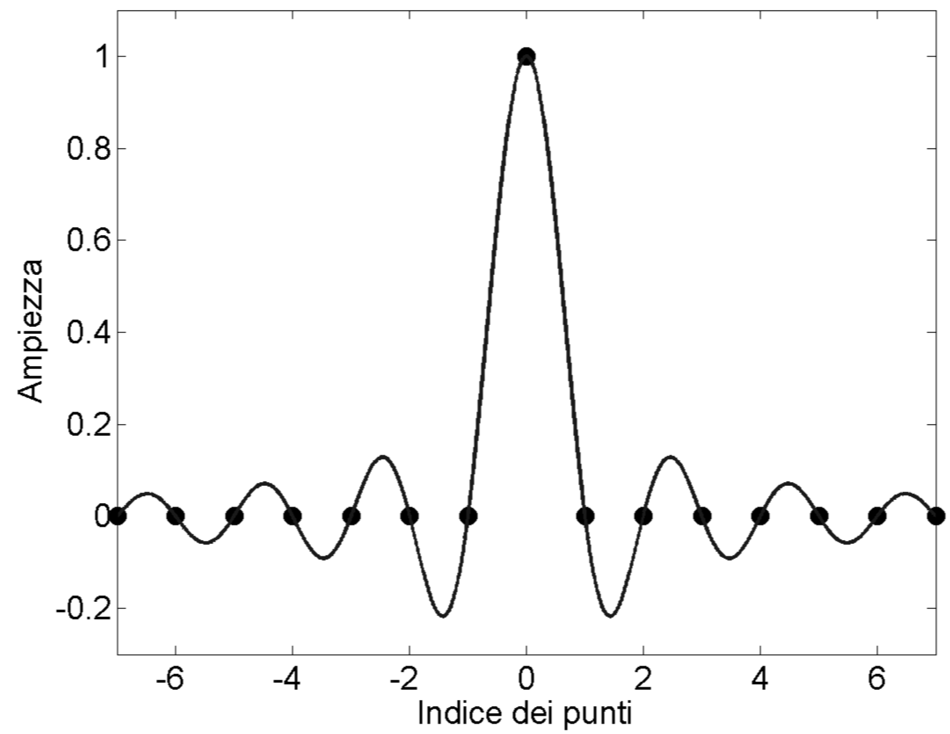
•The interpolation properties, $S(x_i)=f(x_i)$
•The *spline continuity*: $S_{i-1}(x_i) = S_i(x_i)$, i =1,...,n-1
•The continuity of derivatives, $S'_{i-1}(x_i) = S'_i(x_i)$ and $S''_{i-1}(x_i) = S''_i(x_i)$, i =1,...,n -1.

For n polynomials of third degree that form S (so we work on n+1 points), we need to 4n conditions (for each of three polynomial degree there are 4 conditions). Although the interpolation properties gives us n + 1 conditions, the continuity conditions give us n + 1 - 2 = n - 1 conditions, and we get 4n - 2 conditions. We need 2 more conditions that may be of this type:

$S(x_i)'$=*cost* e $S(x_n)'$=cost (bounded spline)

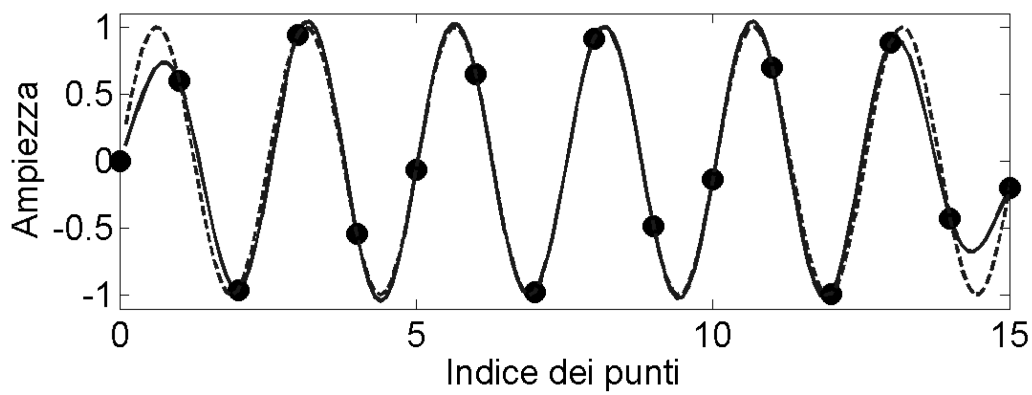
Sinc interpolation

- Used for the reconstruction of signals sampled in time.
- It is derived mathematically from the ideal lowpass filtering of the sampled signal.
- In time domain, it consists of a convolution of the sampled signal with the function: $\mathrm{sinc}(x)=\sin(x)/x$

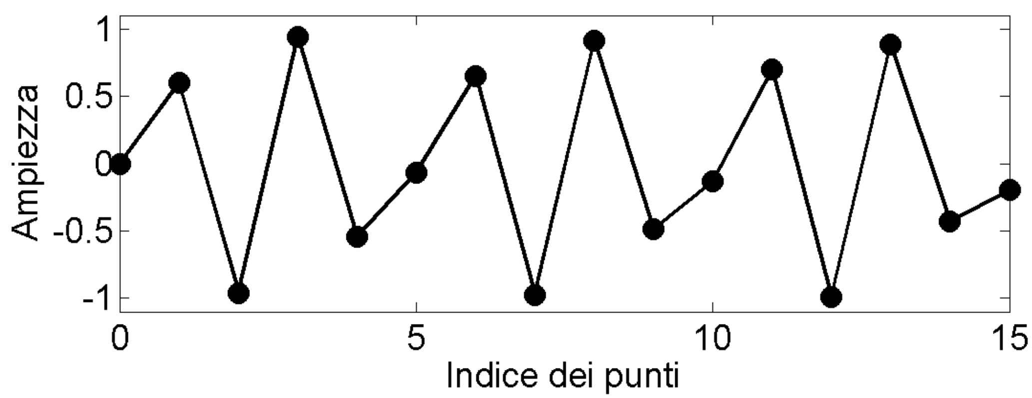


Example of reconstruction of a signal through the interpolator

Sinusoid sampled at 2.51 points per period



Sinc(x) interpolator



Linear interpolator

Fitting of exmperimental points

- An **experimental diagram**, obtained from the measurement results, often show a dependence $y = f(x)$ that appears to be reasonably approximated by a **known function**;
- Alternatively, from a **theoretical analysis**, we can know what kind of **mathematical relationship** should be represented by points, but the dispersion of the data is so large (e.g. for the presence of noise) that we can not define with sufficient **reliability values of the parameters**

- How can I **get these values** (characteristic parameters of the measured phenomenon) **as a measurement / observation of** some points?

Least squares fitting (LS)

- It is considered a generic dependency of a physical variable y from another variable x, through a function f with more parameters $A,B,\ldots$: $y = f(A,B,\ldots x)$
- n measurements (y_i) of variable y as a function of the variable x observed in points x_i are made
- To estimate the parameters that best represent the measured reality, we define a "distance" function between the measure and the function f. The idea is to minimize this distance
- The "distance" function most commonly used is the sum of the standard deviations between f and the measured Difference: $\delta_i = y_i - f(x_i)$
- "Distance" function to be minimized:

$$\Phi = \sum_{i=1}^{n} \delta_i^2$$

Linear fitting LS (1/2)

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An important case of fitting, simple to solve analytically, is that of the linear fitting:

Consider a linear dependence $y = m\,x + q$ and you want to calculate the two parameters m and q.

For the i^{th} measurement point, the deviation between the δ_i empirical value, y_i, and that of the regression curve, $f(x_i)$, that is

$$\delta_i = y_i - [\, m\, x_i + q\,]$$

We need to find the **values of parameters (m and q)** for which a **minimum** is the "**distance**"

$$\Phi(m,q) = \sum_{i=1}^{n} \delta_i^2 = \sum_{i=1}^{n} \left[y_i - \left(m x_i + q \right) \right]^2$$

Linear fitting LS (2/2)

To find the minimum of Φ, we cancel the two first partial derivatives with respect to *m* and *q*:

$$\frac{\partial \Phi}{\partial m} = 0 \Rightarrow \left(m \sum x_i^2 \right) + q \sum x_i = \sum x_i y_i$$

$$\frac{\partial \Phi}{\partial q} = 0 \Rightarrow \left(m \sum x_i \right) + nq = \sum y_i$$

where all summations are obviously extended for *i* ranging from 1 to *n*.

It is obtained a linear system of two equations in two unknowns, *m* and *q* precisely.

Linear fitting: calculation of *m* and *q*

The solution of the system (which is easily obtained by substitution) is:

$$m = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$q = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{\sum y_i - m\sum x_i}{n} = \overline{y} - m\overline{x}$$

This solution corresponds to a minimum (one can mathematically prove by the second derivatives, both> 0, or thinking about the meaning of the "distance" function, inherently positive and growing away from the points acquired ...)

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Exercise about fitting line (1/2)

n(=5) measurement of y=f(x) with experimental points

i	1	2	3	4	5
x_i = [0	1	2	3	4]
y_i = [1	2	2	2	3]

x_i	y_i
0	1
1	2
2	2
3	2
4	3

Modello lineare $\boldsymbol{\delta_i = y_i - [\, m\, x_i + q\,]}$

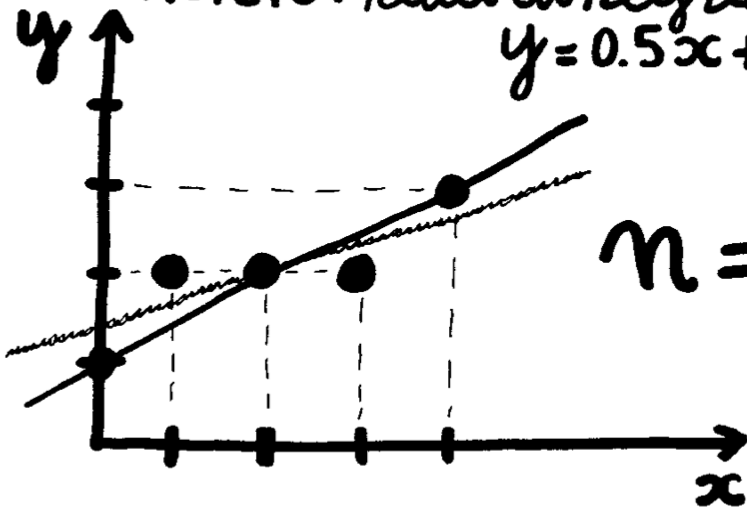
Regressione ai minimi quadrati → $\sum(\delta_i)^2$="min."

$$m = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$q = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \frac{\sum y_i - m\sum x_i}{n} = \overline{y} - m\overline{x}$$

Exercise about fitting line (1/2)

ESERCIZIO: retta di regressione



$n = 5$

x_i	y_i
0	1
1	2
2	2
3	2
4	3

Modello: $y = mx + b$ retta di regr.

$$m = \frac{5(0+2+4+6+12) - 10 \times 10}{5(0+1+4+9+16) - (10)^2} = \frac{20}{50} = 0.4$$

$$b = \frac{10 - 0.4 \times 10}{5} = \frac{6}{5} = 1.2$$