Radiofrequency Measurements

Introduction on Scattering parameters

Measurement of Electromagnetic quantities

- In DC:

  V, I, R, P, E, H

- Low frequency, AC, with an adequate equivalent circuit:

  V, I, Z, P, E, H

- Radio-frequency and microwaves: sometimes *V, I* and *Z* could exist, but not always.

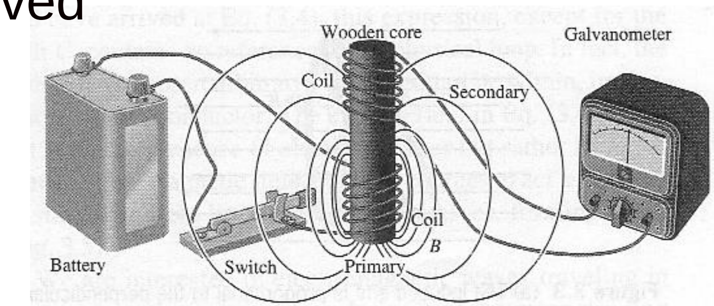
  P, E, H *always exist*

  But we can define reflection and transmission coefficient

  T, Γ *always measureable*

Maxwell's equations

- Based on observation -- not derived

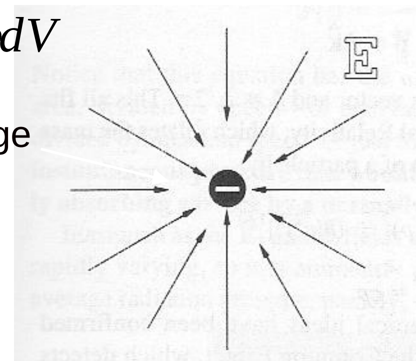


Induction $$\oint_C \vec{E} \bullet d\vec{l} = -\iint_A \frac{\partial \vec{B}}{\partial t} \bullet d\vec{S}$$

Loop voltage — Flux change

Charges give electric field $$\oint_A \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_0} \iiint_V \rho dV$$

Electric field flux — Charge



Capacitor Q = CV

Q = ε A E = V (ε A /d)

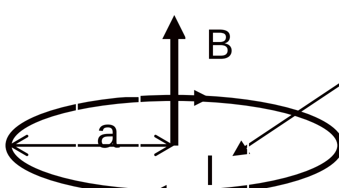
C = ε A /d

I = dQ/dt **= ε A dE/dt**

No magnetic monopoles $$\oint_A \vec{B} \bullet d\vec{S} = 0$$

No net magnetic flux through closed surface

Currents give magnetic field $$\oint_C \vec{B} \bullet d\vec{l} = \mu \iint_A \left(\vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \right) \bullet d\vec{S}$$



Current

Changing electric field

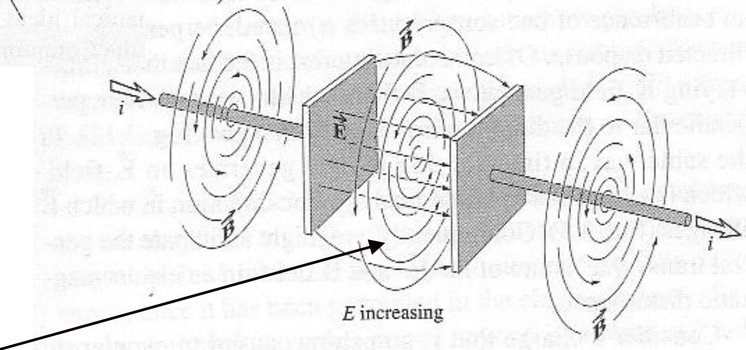


Figure 3.11 (a) Ampère's Law is indifferent to which area A_1 or A_2 is bounded by the path C. Yet a current passes through A_1 and not through A_2, and that means something is very wrong. (b) $\vec{\mathbf{B}}$-field concomitant with a time-varying $\vec{\mathbf{E}}$-field in the gap of a capacitor.

Electromagnetic field in vacuum

- **No sources of electric field, no currents**

$$\oint_C \vec{E}\bullet d\vec{l} = -\iint_A \frac{\partial \vec{B}}{\partial t}\bullet d\vec{S}$$

$$\oint_A \vec{E}\bullet d\vec{S} = 0 \qquad \oint_A \vec{B}\bullet d\vec{S} = 0$$

$$\oint_C \vec{B}\bullet d\vec{l} = \mu_0\varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t}\bullet d\vec{S}$$

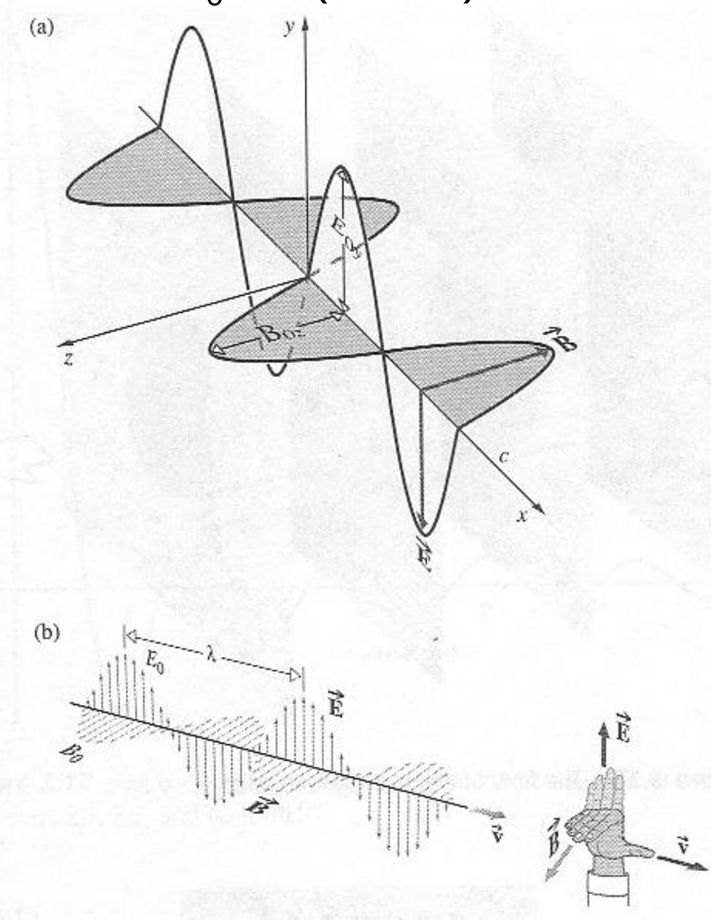
$$\nabla^2 \vec{E} = \varepsilon_0\mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{B} = \varepsilon_0\mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Propagating waves

$E = E_0 \cos(kx - \omega t)$

$B = B_0 \cos(kx - \omega t)$



$$\psi = A \sin k(x \mp vt)$$

$$\psi = A \sin 2\pi \left(\frac{x}{\lambda} \mp \frac{t}{\tau}\right)$$

$$\psi = A \sin 2\pi(\kappa x \mp \nu t)$$

$$\psi = A \sin (kx \mp \omega t)$$

$$\psi = A \sin 2\pi\nu \left(\frac{x}{v} \mp t\right)$$

Light speed: $c = 1/\sqrt{(\mu\varepsilon)} = \omega/k$

$B = E / c$

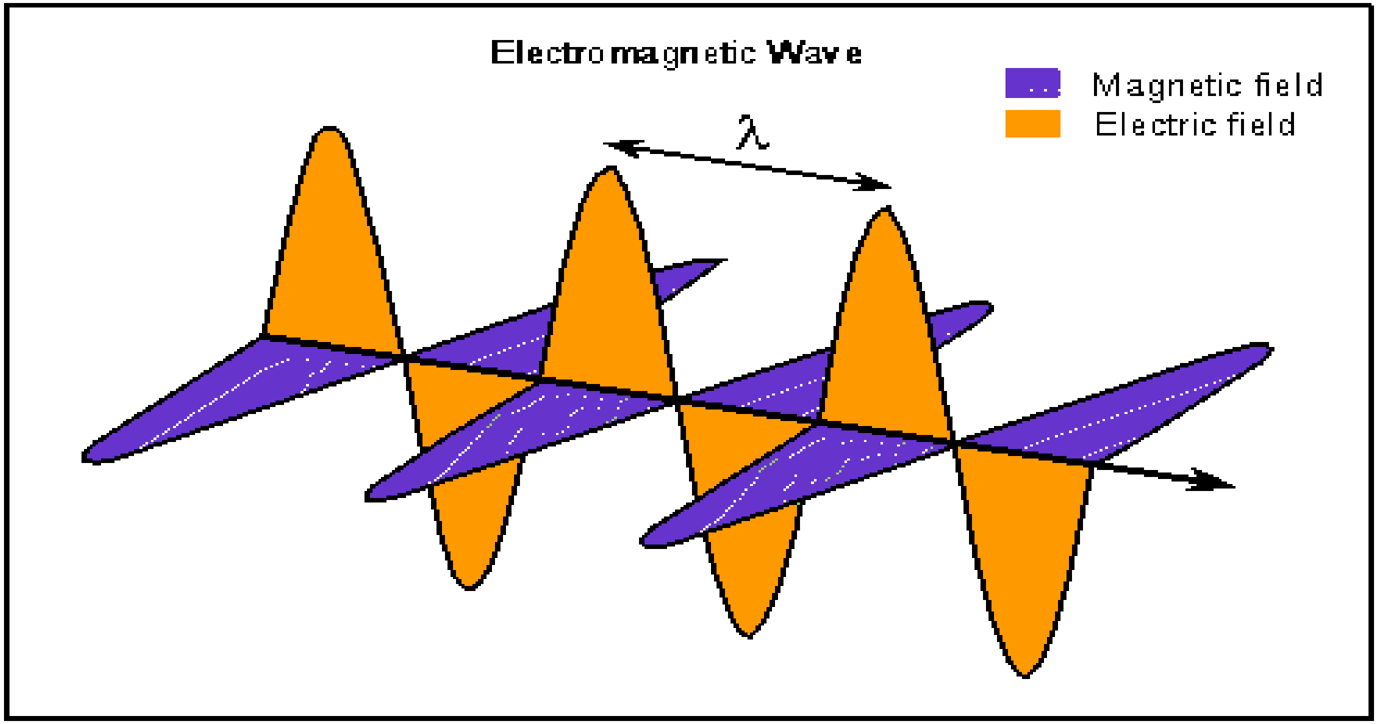
Maxwell's eqns -- differential form

$$\vec{\nabla}\bullet\vec{E} = \rho/\varepsilon_0 \qquad \vec{\nabla}\bullet\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\varepsilon\frac{\partial \vec{E}}{\partial t}$$

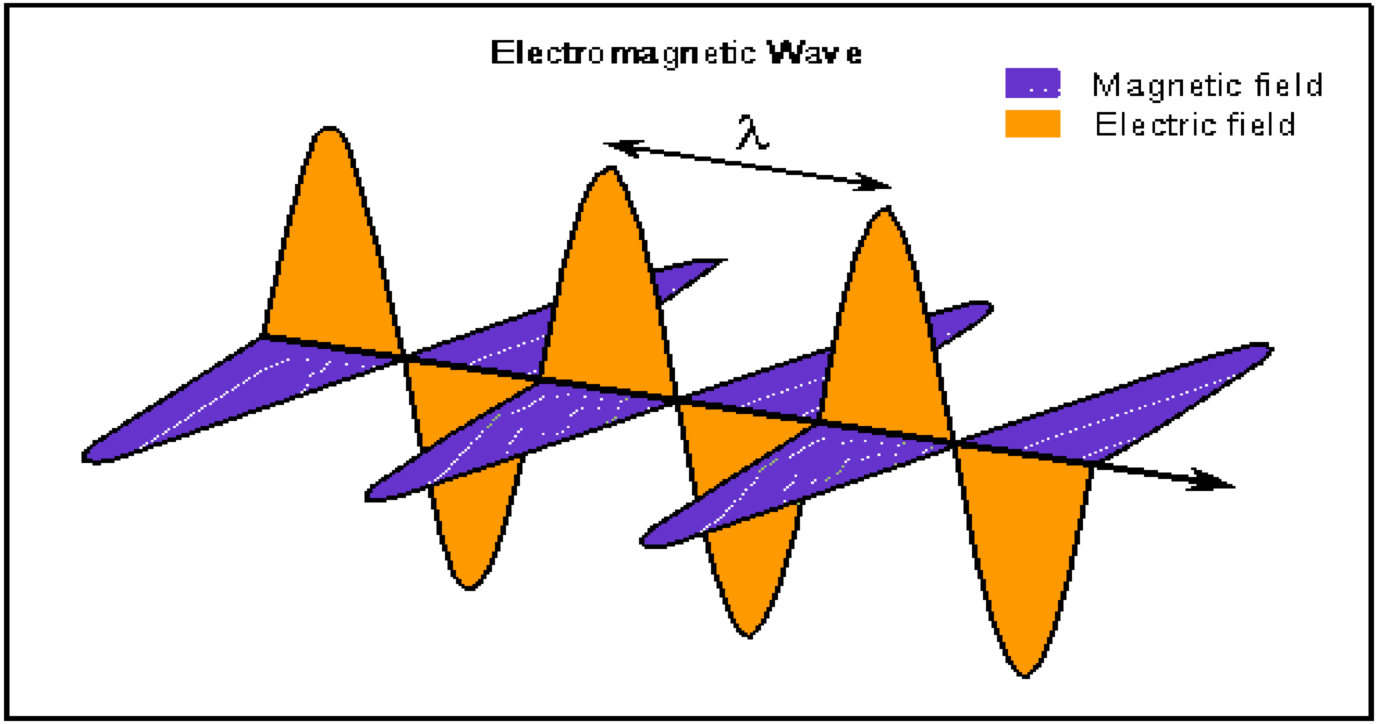
Electromagnetic Waves

- When an electric charge vibrates, the electric field around it changes creating a changing magnetic field.
- The magnetic and electric fields create each other again and again.



Electromagnetic Waves

- An EM wave travels in all directions. The figure only shows a wave traveling in one direction.
- The electric and magnetic fields vibrate at right angles to the direction the wave travels so it is a transverse wave.



Electromagnetic Field Propagation

In a guided electromagnetic wave (propagating through Z) the transverse fields (one propagation mode) can be written as

$$\vec{E}_t = v(z) \cdot \vec{e}(x, y) \qquad \vec{H}_t = i(z) \cdot \vec{h}(x, y)$$

the first terms are complex numbers, named voltage and current "generalized", with dimensions of voltage and current

the vectors describe the field distribution in the XY plane (transversal) and do not depend on Z

There is ambiguity in the definition, but it can be fixed by two normalization:

Power Impedance

The goal is to obtain v = real voltage i = real current, when possible

Electromagnetic Field Propagation

Power normalization - Poynting theorem:

$$P = \frac{1}{2}\mathrm{Re}\left[\int_S \left(\vec{E}_t \times \vec{H}_t^* \cdot \vec{e}_z\right) ds\right] = \frac{1}{2}\mathrm{Re}\left[v \cdot i^*\right] \cdot \int_S \left(\vec{e} \times \vec{h} \cdot \vec{e}_z\right) \cdot ds = \frac{1}{2}\mathrm{Re}\left[v \cdot i^*\right] \cdot W_0$$

Impedance normalization

Each mode-field is due to the sum of the propagating wave and the anti-propagating wave (same spatial distribution, different orientation of E and H)

$$\vec{E}_t = \vec{E}_t^+ + \vec{E}_t^- \qquad v = v^+ + v^-$$

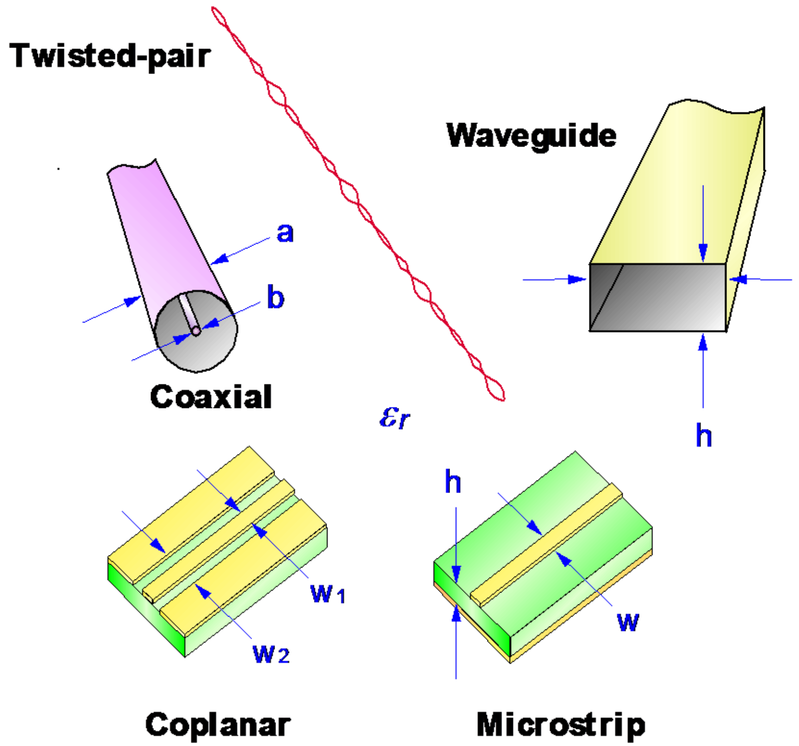
$$\vec{H}_t = \vec{H}_t^+ - \vec{H}_t^- \qquad i = i^+ - i^-$$

If we consider only a propagating wave, the electric and magnetic field are related by the characteristic impedance Z_0

$$Z_w = \frac{E_t^+}{H_t^+} = \frac{v^+}{i^+} \cdot \frac{e(x,y)}{h(x,y)} = Z_0 \cdot z_w \qquad \left(Z_0 = \frac{v^+}{i^+}\right)$$

Transmission line Zo

- Zo determines relationship between voltage and current waves
- Zo is a function of physical dimensions and ε_r
- Zo is usually a real impedance (e.g. 50 or 75 ohms)



Pseudo-waves

We can introduce two complex numbers for describing the electromagnetic wave: the propagating waves a and b

Definitions:

$$\frac{v}{\sqrt{Z_0}} = a + b \qquad \sqrt{Z_0} \cdot i = a - b$$

$$a = \frac{1}{2}\left(\frac{v}{\sqrt{Z_0}} + \sqrt{Z_0} \cdot i\right) = \frac{v^+}{\sqrt{Z_0}} \qquad b = \frac{1}{2}\left(\frac{v}{\sqrt{Z_0}} - \sqrt{Z_0} \cdot i\right) = \frac{v^-}{\sqrt{Z_0}}$$

$$P = \frac{1}{2}\mathrm{Re}\left[v \cdot i^*\right] = \frac{1}{2}\left(|a|^2 - |b|^2\right) = P(+z) - P(-z)$$

Pseudo-waves

The absolute values of a and b indicate "the amplitude":

$$P(+z) = \frac{|a|^2}{2}$$

$$P(-z) = \frac{|b|^2}{2}$$

The phases of a and b are exactly the phases of the electric field *E*

Their ratio indicates the reflection coefficient:

$$\Gamma = \frac{v^-}{v^+} = \frac{b}{a} = \frac{v - Z_0 \cdot i}{v + Z_0 \cdot i} = \frac{Z - Z_0}{Z + Z_0}$$

$$Z = \frac{v}{i}$$

Reflection Coefficient

The characteristic impedance Z_0 is

$$Z_0 = \frac{v^+}{i^+} = \frac{v^-}{i^-}$$

The Reflection coefficient is

$$\Gamma = \frac{v^-}{v^+}$$

The “real” impedance Z is

$$Z = \frac{V}{I}$$

where V and I are the “real” voltage and current

On a real conductor $V = v^+ + v^-$ $I = i^+ - i^-$

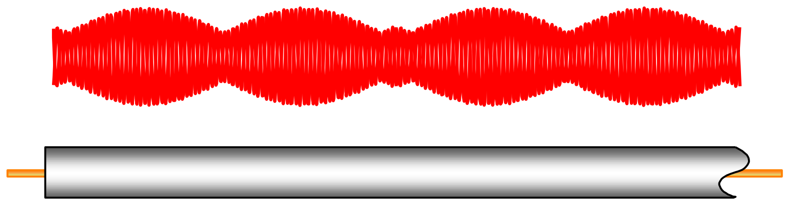
$$Z = \frac{V}{I} = \frac{v^+ + v^-}{i^+ - i^-} = Z_0 \frac{v^+ + v^-}{v^+ - v^-} = Z_0 \frac{1 + \frac{v^-}{v^+}}{1 - \frac{v^-}{v^+}} = Z_0 \frac{1 + \Gamma}{1 + \Gamma}$$

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

Transmission Line Basics

Low frequencies

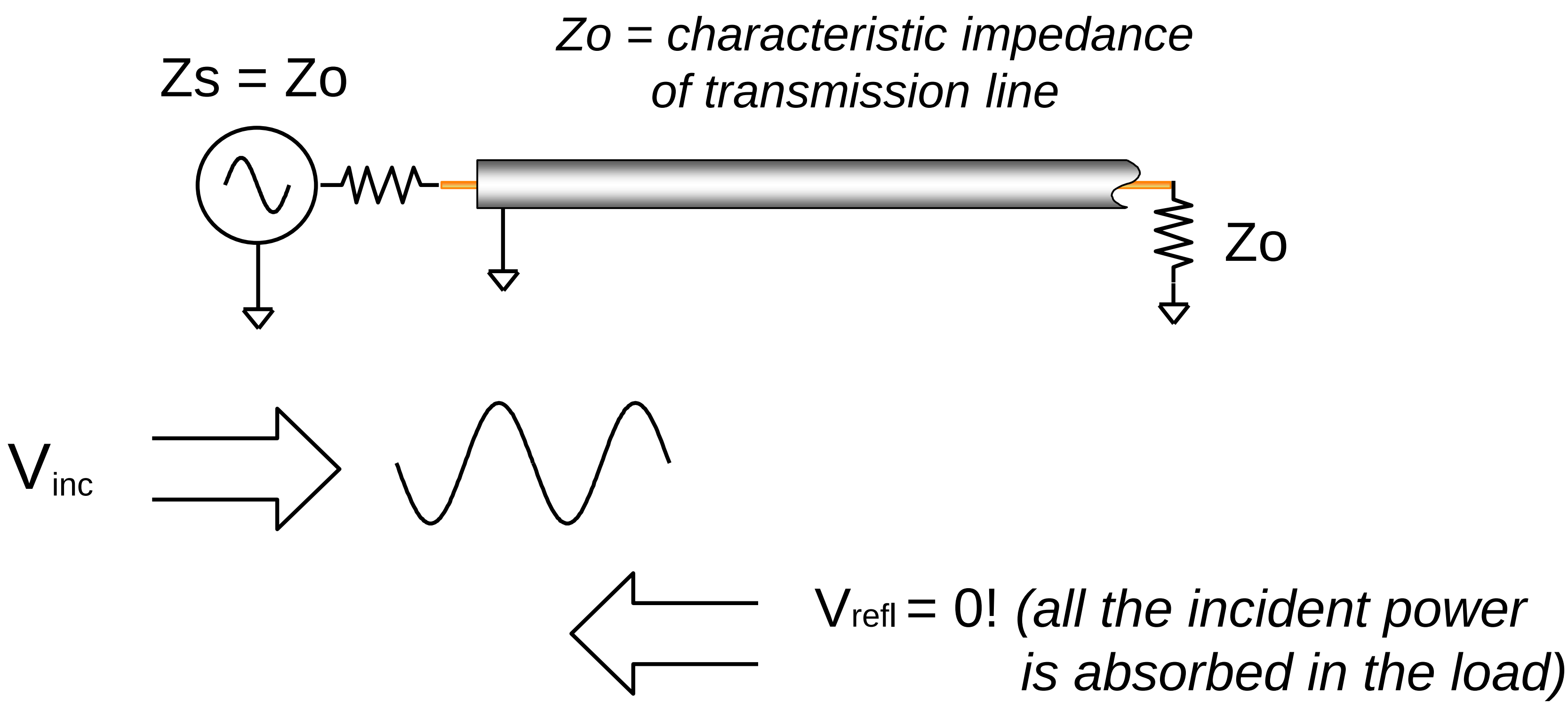
- wavelengths >> wire length
- current (I) travels down wires easily for efficient power transmission
- measured voltage and current not dependent on position along wire



High frequencies

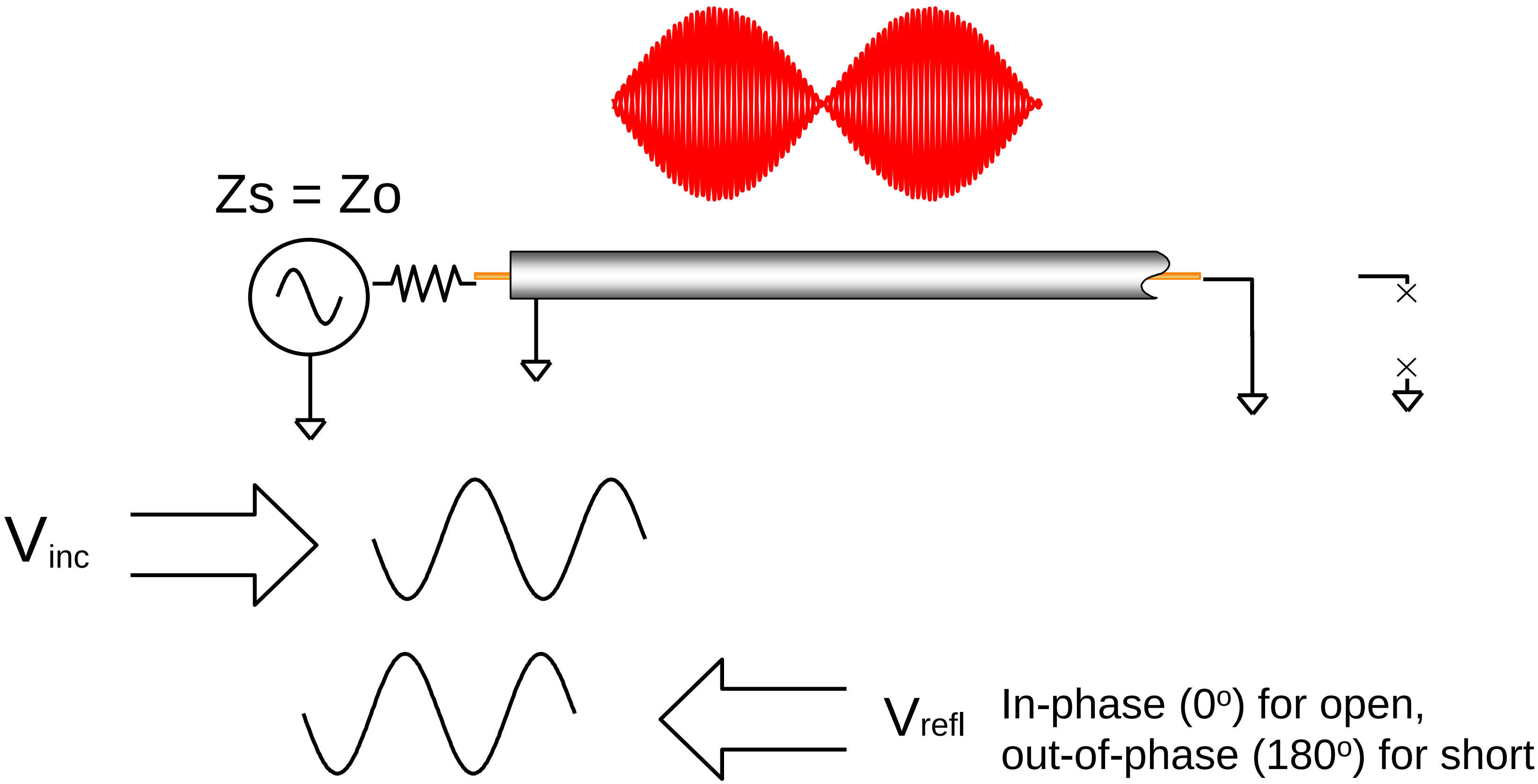
- wavelength ≈ or << length of transmission medium
- need transmission lines for efficient power transmission
- matching to characteristic impedance (Z_o) is very important for low reflection and maximum power transfer
- measured envelope voltage dependent on position along line

Transmission Line Terminated with Zo



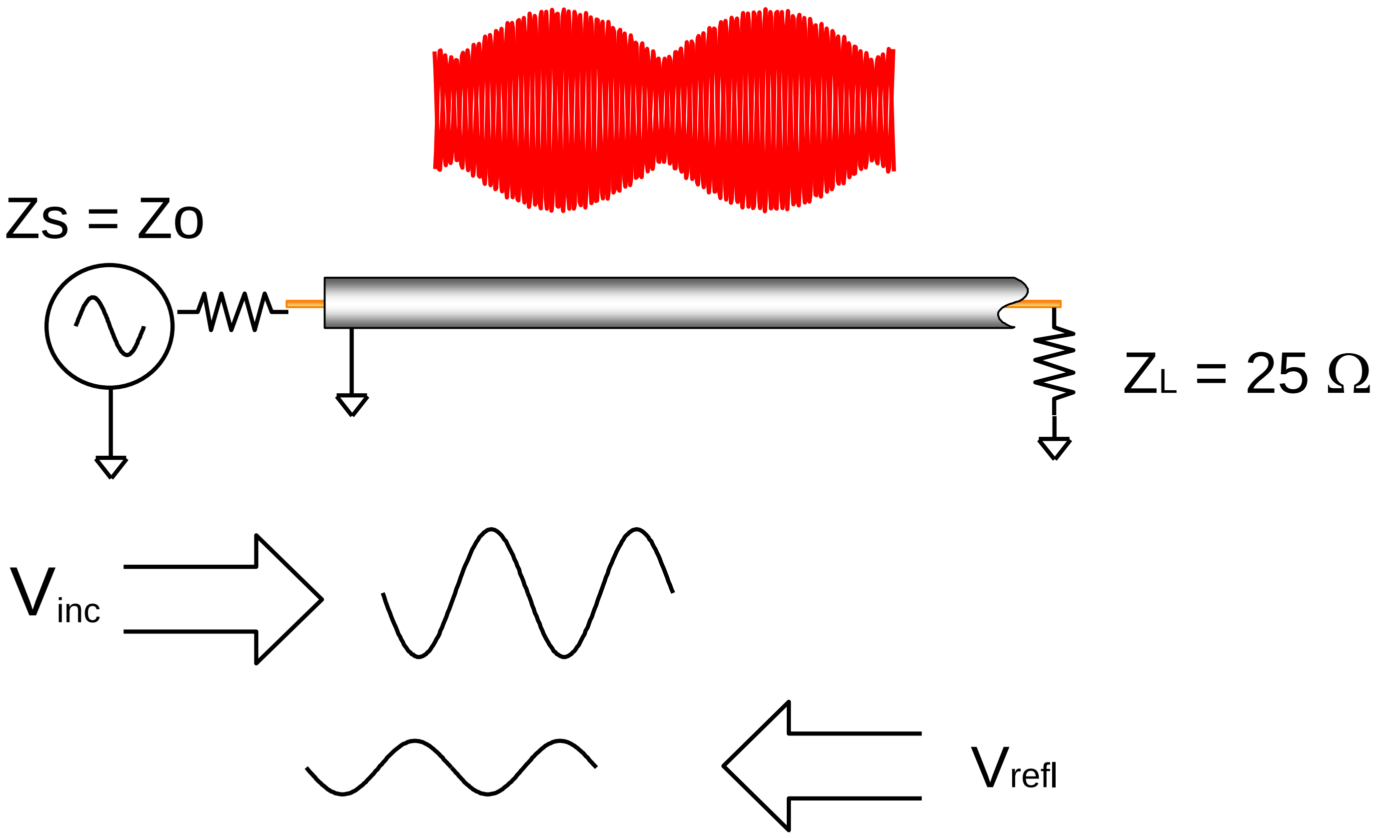
For reflection, a transmission line terminated in Zo behaves like an infinitely long transmission line

Transmission Line Terminated with Short, Open



For reflection, a transmission line terminated in a short or open reflects all power back to source

Transmission Line Terminated with 25 Ω



Standing wave pattern does not go to zero as with short or open

Network described by a Matrix….

Impedance Matrix:

$$\begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1n} \\ Z_{21} & Z_{22} & \dots & Z_{2n} \\ \dots & \dots & \dots & \dots \\ Z_{n1} & Z_{n2} & \dots & Z_{nn} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ \dots \\ i_n \end{bmatrix}$$

It is easy that some terms do not exists… also v and I could not exist..

It is the same for the admittance matrix…

New definition: the SCATTERING MATRIX *S*

$$[b] = [S] \cdot [a]$$

$$\begin{cases} b_1 = S_{11}a_1 + S_{12}a_2 + \dots + S_{1n}a_n \\ b_2 = S_{21}a_1 + S_{22}a_2 + \dots + S_{2n}a_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_n = S_{n1}a_1 + S_{n2}a_2 + \dots + S_{nn}a_n \end{cases}$$

Scattering Matrix

$$\begin{cases} b_1 = S_{11}a_1 + S_{12}a_2 + \ldots + S_{1n}a_n \\ b_2 = S_{21}a_1 + S_{22}a_2 + \ldots + S_{2n}a_n \\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots \\ b_n = S_{n1}a_1 + S_{n2}a_2 + \ldots + S_{nn}a_n \end{cases}$$

a = input waves

b = output waves

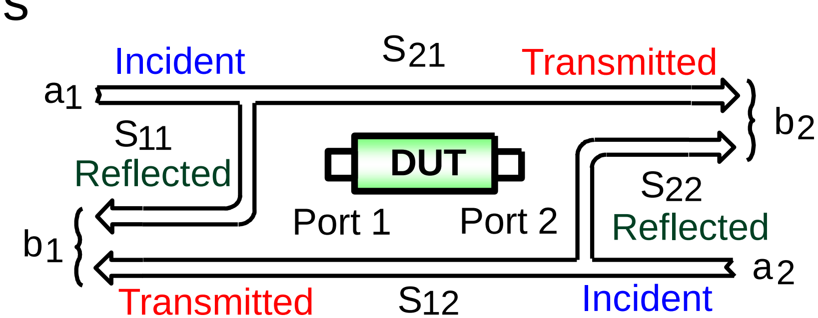
$b_i = S_{ik}a_k$ when $a_h = 0 \ \forall h \neq k$ → $S_{ik} = \dfrac{b_i}{a_k}$ transmission coefficients

Matched port

$b_k = S_{kk}a_k$ when $a_h = 0 \ \forall h \neq k$ → $S_{kk} = \dfrac{b_k}{a_k}$ reflection coefficients

Scattering parameters

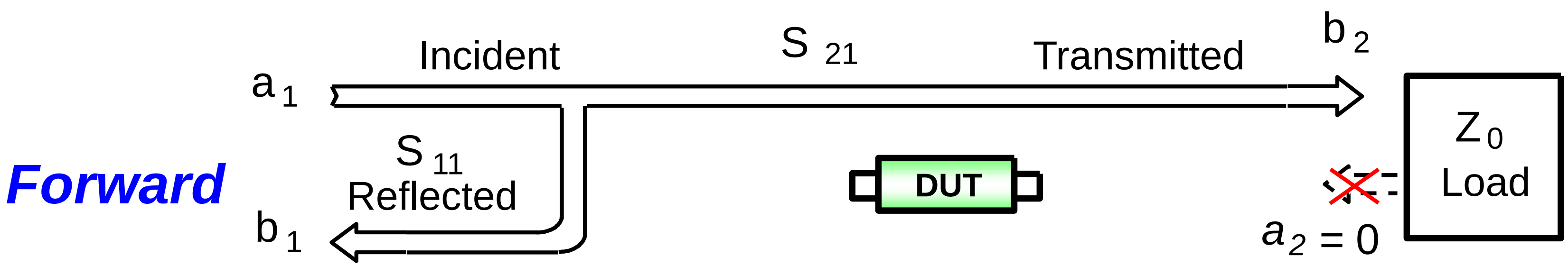
- relatively easy to **obtain** at high frequencies
  - measure voltage traveling waves with a vector network analyzer
  - don't need shorts/opens which can cause active devices to oscillate or self-destruct
- relate to **familiar** measurements (gain, loss, reflection coefficient ...)
- can **cascade** S-parameters of multiple devices to predict system performance
- can **compute** H, Y, or Z parameters from S-parameters if desired
- can easily import and use S-parameter files in our **electronic-simulation** tools



$$b_1 = S_{11} a_1 + S_{12} a_2$$

$$b_2 = S_{21} a_1 + S_{22} a_2$$

Scattering parameters: two-port

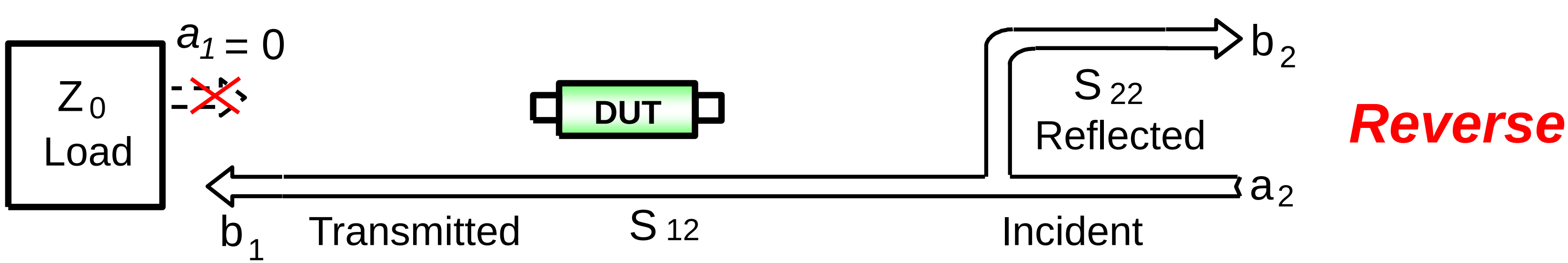


$$S_{11} = \frac{Reflected}{Incident} = \frac{b_1}{a_1}\Bigg|_{a_2 = 0}$$

$$S_{21} = \frac{Transmitted}{Incident} = \frac{b_2}{a_1}\Bigg|_{a_2 = 0}$$

$$S_{22} = \frac{Reflected}{Incident} = \frac{b_2}{a_2}\Bigg|_{a_1 = 0}$$

$$S_{12} = \frac{Transmitted}{Incident} = \frac{b_1}{a_2}\Bigg|_{a_1 = 0}$$



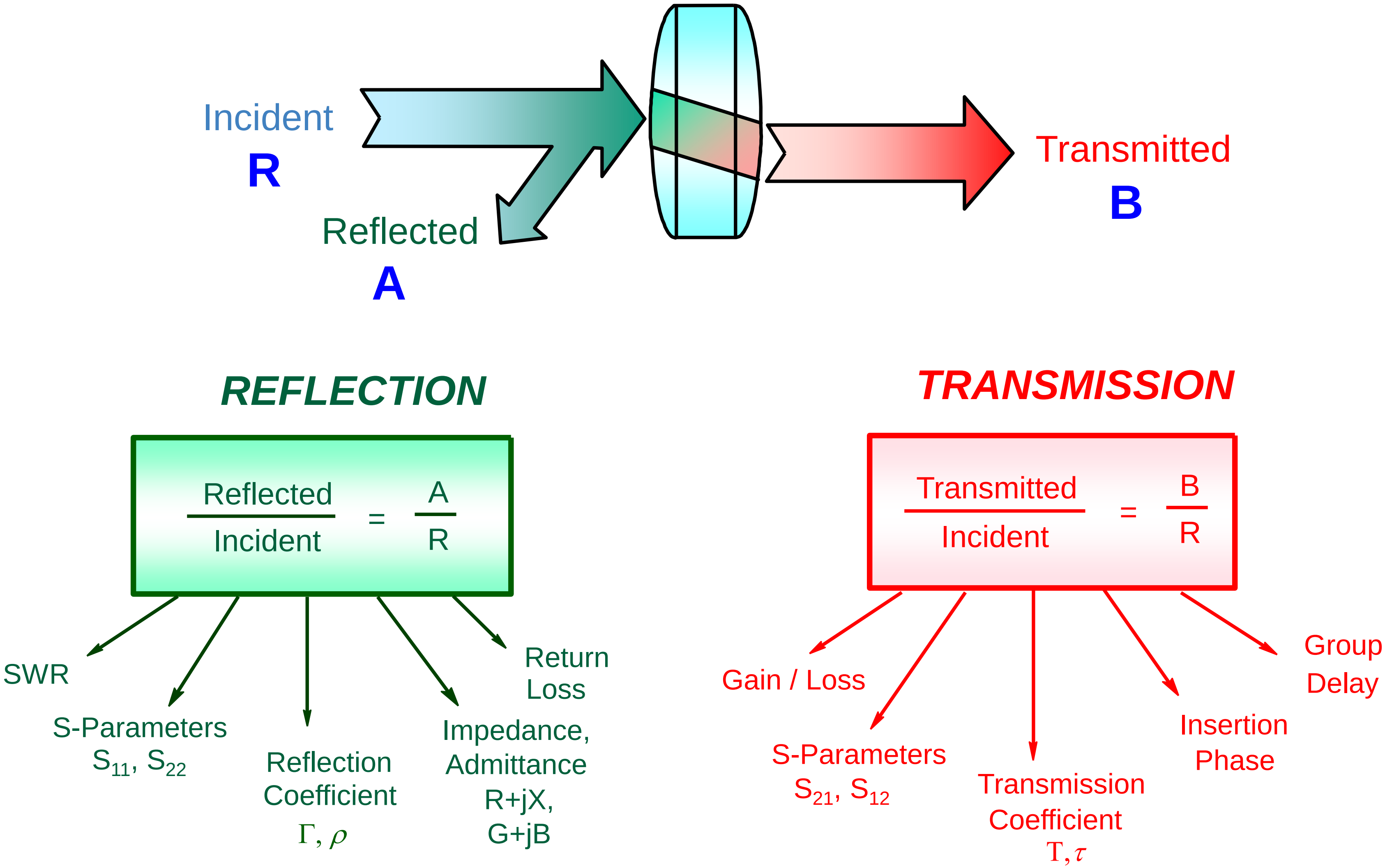
Equating S-Parameters with Common Measurement Terms

S_{11} = forward reflection coefficient ***(input match)***
S_{22} = reverse reflection coefficient ***(output match)***
S_{21} = forward transmission coefficient ***(gain or loss)***
S_{12} = reverse transmission coefficient ***(isolation)***

S-parameters are inherently complex, linear quantities

we often express them in a log-magnitude format

High-Frequency Device Characterization



Propagation

A wave is described as $\sin(\omega t \pm \beta z) = \sin[\Phi(t)]$

The wavefront $\Phi = \Phi_0$ propagates in the versus depending on the sign

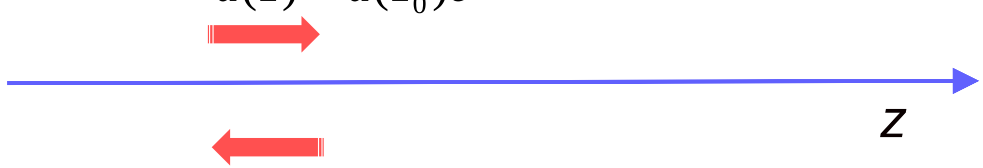
$\sin(\omega t + \beta z)$ propagates towards (-z)

$\sin(\omega t - \beta z)$ propagates towards (+z)

Time period $T = \dfrac{2\pi}{\omega}$ Spatial period $\lambda = \dfrac{2\pi}{\beta}$

In complex notation (the time dependence is implicit):

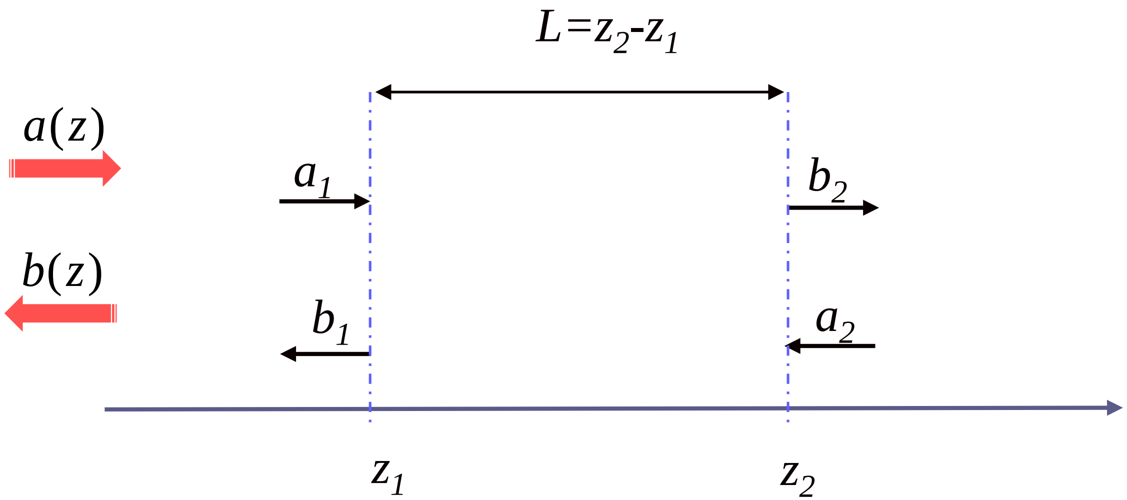
$$a(z) = a(z_0)e^{-j\beta(z-z_0)}$$



$$b(z) = b(z_0)e^{j\beta(z-z_0)}$$

Scattering matrix of a transmission line

Transmission line with length *L* LOSSLESS



$$a(z_2) = a(z_1) e^{-j\beta(z_2 - z_1)}$$
$$b(z_2) = b(z_1) e^{j\beta(z_2 - z_1)}$$

$$b_2 = a_1 e^{-j\beta L}$$
$$b_1 = a_2 e^{-j\beta L}$$

$$S_{21} = e^{-j\beta L}$$
$$S_{12} = e^{-j\beta L}$$

$$S = \begin{bmatrix} 0 & e^{-j\beta L} \\ e^{-j\beta L} & 0 \end{bmatrix}$$

Loss and Reflection Coefficient

Transmission line with length *L*

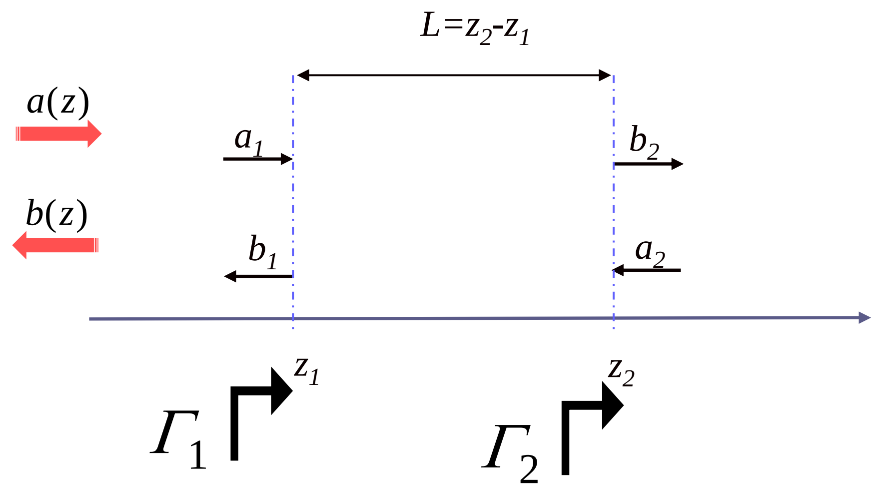
WITH LOSS

$$\gamma = \alpha + j\beta$$

$$b_2 = a_1 e^{-\gamma L}$$
$$b_1 = a_2 e^{-\gamma L}$$

$$S_{21} = e^{-\gamma L}$$
$$S_{12} = e^{-\gamma L}$$

$$S = \begin{bmatrix} 0 & e^{-\gamma L} \\ e^{-\gamma L} & 0 \end{bmatrix}$$



$$\Gamma_1 = \frac{b_1}{a_1} = \frac{a_2 e^{-\gamma l}}{b_2 e^{\gamma l}} = \Gamma_2 e^{-2\gamma l}$$

$$\Gamma = |\Gamma| e^{j\varphi}$$

$$|\Gamma_1| = |\Gamma_2| e^{-2\alpha l}$$

$$\varphi_1 = \varphi_2 - 2\beta L + 2n\pi$$

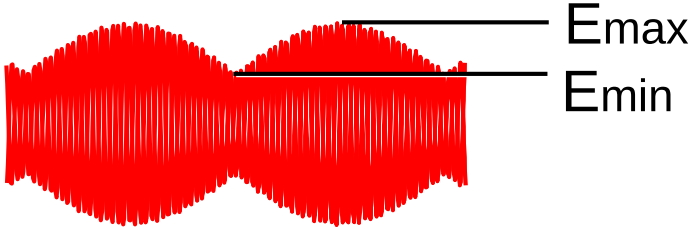
Reflection Parameters

Reflection Coefficient

$$\Gamma = \frac{V_{reflected}}{V_{incident}} = \rho \angle \Phi = \frac{Z_L - Z_O}{Z_L + Z_O}$$

Return loss = -20 log(ρ), $\rho = |\Gamma|$

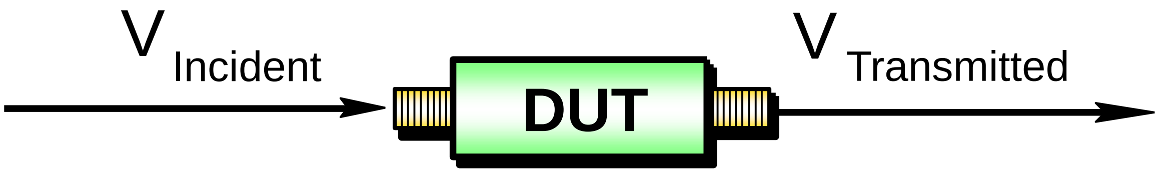
Voltage Standing Wave Ratio (VSWR)



$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{|a|+|b|}{|a|-|b|} = \frac{v_{max}}{v_{min}} = \frac{E_{max}}{E_{min}}$$

No reflection ($Z_L = Zo$)		***Full reflection (Z_L = open, short)***
0	ρ	1
∞ dB	**RL**	0 dB
1	**VSWR**	∞

Transmission Parameters



$$\text{Transmission Coefficient} = T = \frac{V_{\text{Transmitted}}}{V_{\text{Incident}}} = \tau\angle\phi$$

$$\text{Insertion Loss (dB)} = -20 \text{ Log} \left| \frac{V_{\text{Trans}}}{V_{\text{Inc}}} \right| = -20 \log \tau$$

$$\text{Gain (dB)} = 20 \text{ Log} \left| \frac{V_{\text{Trans}}}{V_{\text{Inc}}} \right| = 20 \log \tau$$

Example of scattering matrix: RF amplifier

A 50 Ω microwave integrated circuit (MIC) amplifier has the following *s*-parameters:

$$s_{11} = 0.12 \angle -10^\circ \qquad s_{12} = 0.002 \angle -78^\circ$$
$$s_{21} = 9.8 \angle 160^\circ \qquad s_{22} = 0.01 \angle -15^\circ$$

Calculate: (a) input VSWR, (b) return loss, (c) forward insertion power gain and (d) reverse insertion power loss.

Given: $s_{11} = 0.12 \angle -10^\circ \qquad s_{12} = 0.002 \angle -78^\circ$
$s_{21} = 9.8 \angle 160^\circ \qquad s_{22} = 0.01 \angle -15^\circ$

Required: (a) Input VSWR, (b) return loss, (c) forward insertion power gain, (d) reverse insertion power loss.

Example of scattering matrix: RF amplifier

Solution

(a) From Equation 2.38

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+|s_{11}|}{1-|s_{11}|} = \frac{1+0.12}{1-0.12}$$
$$= 1.27$$

(b) Return loss (dB) = $-20 \log_{10} 0.12 = 18.42$ dB

(c) Forward insertion gain = $|s_{21}|^2 = (9.8)^2 = 96.04$ or

Forward insertion gain = $10 \log_{10} (9.8)^2$ dB = 19.83 dB

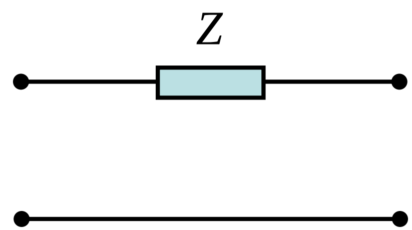
(d) Reverse insertion gain = $|s_{12}|^2 = (0.002)^2 = 4 \times 10^{-6}$ or

Reverse insertion gain = $10 \log_{10} (0.002)^2$ dB = –53.98 dB

The amplifier is virtually unilateral with (53.98 – 19.83) or 34.15 dB of output to input isolation.

S-parameters of a series impedance

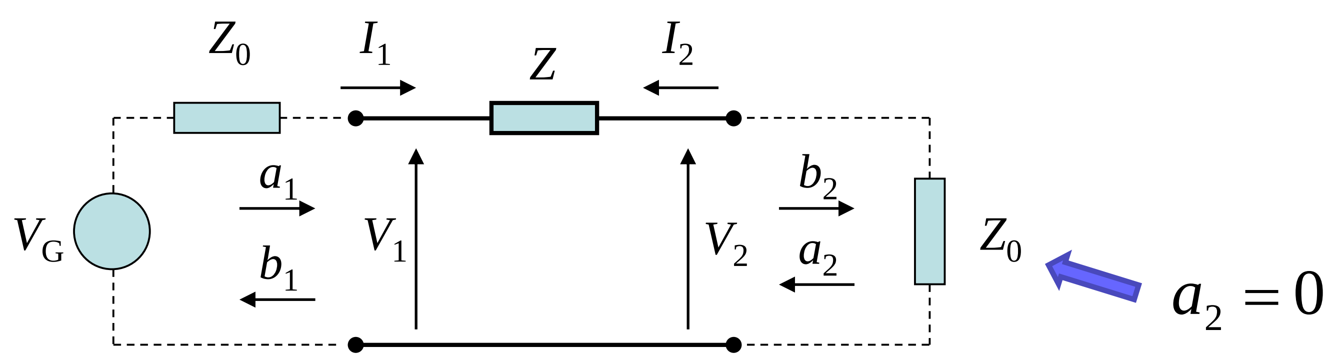
Find the S-parameters of a series impedance Z connected between the two ports



$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

We can apply the definitions of the pseudo-waves, and solve the simple circuit.

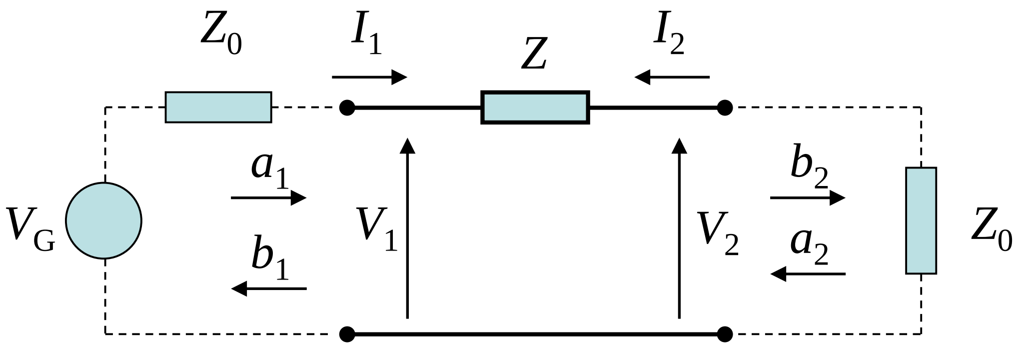
$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \qquad a_1 = \frac{1}{2}\left(\frac{V_1}{\sqrt{Z_0}} + \sqrt{Z_0}\cdot I_1\right) \qquad b_1 = \frac{1}{2}\left(\frac{V_1}{\sqrt{Z_0}} - \sqrt{Z_0}\cdot I_1\right)$$



matched source

matched load

S-parameters of a series impedance



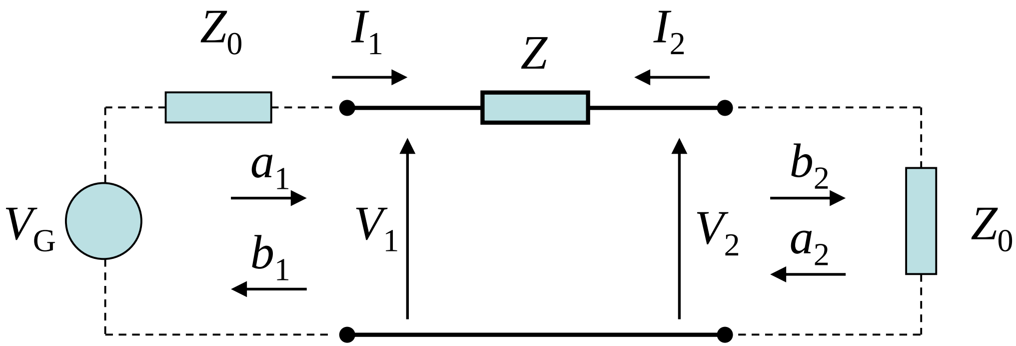
$$V_1 = \frac{Z+Z_0}{Z+2Z_0} V_G \qquad V_2 = \frac{Z_0}{Z+2Z_0} V_G \qquad I_1 = \frac{V_G}{Z+2Z_0} = -I_2$$

$$a_1 = \frac{1}{2}\left(\frac{V_1}{\sqrt{Z_0}} + \sqrt{Z_0}\cdot I_1\right) = \frac{1}{2}\left(\frac{Z+Z_0}{Z+2Z_0}\frac{V_G}{\sqrt{Z_0}} + \sqrt{Z_0}\cdot\frac{V_G}{Z+2Z_0}\right) = \frac{V_G}{2\sqrt{Z_0}}$$

$$b_1 = \frac{1}{2}\left(\frac{V_1}{\sqrt{Z_0}} - \sqrt{Z_0}\cdot I_1\right) = \frac{1}{2}\left(\frac{Z+Z_0}{Z+2Z_0}\frac{V_G}{\sqrt{Z_0}} - \sqrt{Z_0}\cdot\frac{V_G}{Z+2Z_0}\right) = \frac{V_G}{2\sqrt{Z_0}}\frac{Z}{Z+2Z_0}$$

$$a_2 = \frac{1}{2}\left(\frac{V_2}{\sqrt{Z_0}} + \sqrt{Z_0}\cdot I_2\right) = 0 \quad \text{as expected}$$

S-parameters of a series impedance



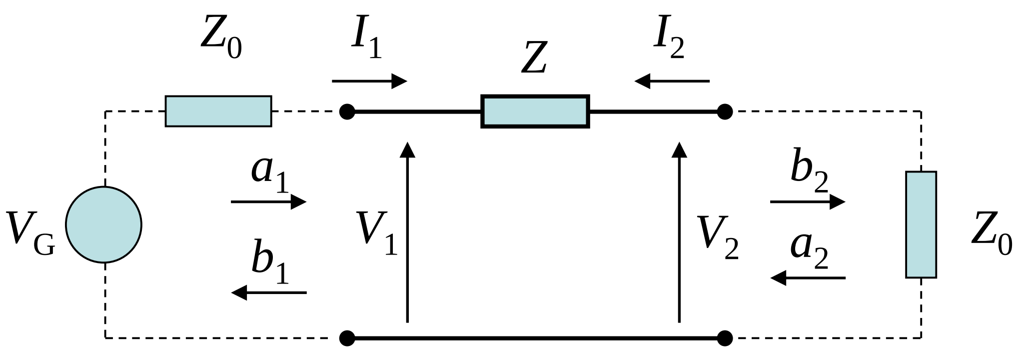
$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{Z}{Z+2Z_0}$$

$$= \Gamma_1 = \frac{Z_{IN} - Z_0}{Z_{IN} + Z_0} \quad \text{with} \quad Z_{IN} = Z + Z_0$$

$$b_2 = \frac{1}{2}\left(\frac{V_2}{\sqrt{Z_0}} - \sqrt{Z_0}\cdot I_2\right) = \frac{1}{2}\left(\frac{Z_0}{Z+2Z_0}\frac{V_G}{\sqrt{Z_0}} - \sqrt{Z_0}\cdot\left(-\frac{V_G}{Z+2Z_0}\right)\right) = \frac{V_G}{2\sqrt{Z_0}}\frac{2Z}{Z+2Z_0}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \frac{2Z_0}{Z+2Z_0}$$

S-parameters of a series impedance



It is possible to repeat the calculation feeding the signal to the port 2, and closing the port 1. For symmetry we get:

$$S_{22} = S_{11} = \frac{Z}{Z + 2Z_0} \qquad S_{12} = S_{21} = \frac{2Z_0}{Z + 2Z_0}$$

$$S = \begin{bmatrix} \dfrac{Z}{Z + 2Z_0} & \dfrac{2Z_0}{Z + 2Z_0} \\ \dfrac{2Z_0}{Z + 2Z_0} & \dfrac{Z}{Z + 2Z_0} \end{bmatrix}$$

Scattering matrix properties: LOSSLESS

If the network is lossless, the input power is equal to the output power

$$P_{IN} = \sum_{i=1}^{n} \frac{|a_i|^2}{2} \qquad = P_{OUT} = \sum_{i=1}^{n} \frac{|b_i|^2}{2} \quad \Longrightarrow \quad \sum_{i=1}^{n} |b_i|^2 - \sum_{i=1}^{n} |a_i|^2 = 0$$

$$|a_i|^2 = a_i \cdot a_i^*$$

a_i^* indicates the complex coniugate

$$\sum_{i=1}^{n} |b_i|^2 = [b_1, b_2, \ldots\ldots, b_n] \begin{bmatrix} b_1^* \\ b_2^* \\ \cdots \\ b_n^* \end{bmatrix} = \overline{[b]}[b]^*$$

$\overline{[b]}$ indicates the transposed matrix - vector

$$[b] = [S] \cdot [a] \quad \Longrightarrow \quad \sum_{i=1}^{n} |b_i|^2 = \overline{[b]}[b]^* = \overline{[S][a]}[S]^*[a]^*$$

Lets substitute the matrix formalism in the first balancing equation

Scattering matrix properties: LOSSLESS

Lets substitute the matrix formalism in the first balancing equation

$$\sum_{i=1}^{n} |a_i|^2 = \overline{[a]}[a]^* \qquad \sum_{i=1}^{n} |b_i|^2 = \overline{[b]}[b]^* = \overline{[S][a]}[S]^*[a]^*$$

$$\Downarrow$$

$$\sum_{i=1}^{n} |b_i|^2 - \sum_{i=1}^{n} |a_i|^2 = 0 = \overline{[S]\cdot[a]}\cdot[S]^*\cdot[a]^* - \overline{[a]}\cdot[a]^* =$$

$$\overline{[a]}\cdot\overline{[S]}\cdot[S]^*\cdot[a]^* - \overline{[a]}\cdot[I]\cdot[a]^* = \overline{[a]}\cdot\left(\overline{[S]}\cdot[S]^* - [I]\right)\cdot[a]^* = 0$$

$$\forall a \Rightarrow \quad \overline{[S]}\cdot[S]^* = [I] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

The matrix with this property is named **unitary matrix**

Scattering matrix properties: LOSSLESS

The scattering matrix of a lossless network is UNITARY:

$$\overline{[S]} \cdot [S]^* = [I] \quad \Rightarrow \quad \left|\det[S]\right| = 1$$

The network described by a unitary matrix is lossless (the output power is equal to the input power)

Example: 2x2 unitary matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

No loss for the power entering the port 1

$$\overline{S} \cdot S^* = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} \left|S_{11}\right|^2 + \left|S_{21}\right|^2 = 1 \\ S_{11} S_{12}^* + S_{21} S_{22}^* = 0 \\ S_{12} S_{11}^* + S_{22} S_{21}^* = 0 \\ \left|S_{12}\right|^2 + \left|S_{22}\right|^2 = 1 \end{cases}$$

No loss for the power entering the port 2

Scattering matrix properties: RECIPROCITY

The network **reciprocity** is a strong physical property, always satisfied if there are **no anisotropic elements**

For Impedance and admittance matrix the property is: $Z_{ik} = Z_{ki}$; $Y_{ik} = Y_{ki}$

Given the characteristic impedance Z_{0i} for port I, the reciprocity condition for a scattering matrix is

$$Z_{0i}^{-1} \cdot S_{ik} = S_{ki} \cdot Z_{0k}^{-1}$$

If the access guide have the same characteristic impedance, the condition becomes

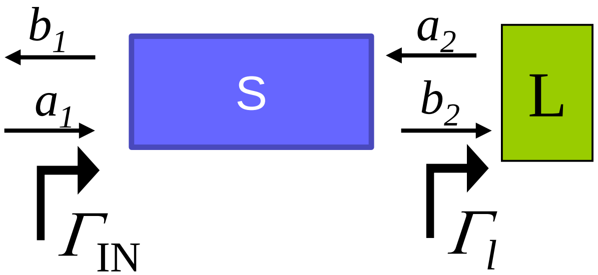
$$S_{ik} = S_{ki}$$

The scattering matrix of a reciprocal network is symmetric.

Reflection Coefficient

Let's consider a load L, with reflection coefficient Γ_l

What is the reflection coefficient value, Γ_{IN} , seen at the input of the network shown in figure?



$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$
$$\Gamma_l = \frac{a_2}{b_2}$$

$$\Gamma_{IN} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l} = \frac{S_{11} + (S_{12}S_{21} - S_{11}S_{22})\Gamma_l}{1 - S_{22}\Gamma_l}$$

The reflection coefficient changes following a bilinear relation

$$\Gamma_{IN} = \frac{S_{11} + (S_{12}S_{21} - S_{11}S_{22})\Gamma_l}{1 - S_{22}\Gamma_l} = \frac{A\Gamma_l + B}{C\Gamma_l + D}$$

Properties of a bilinear relation

Consider two complex variables *w* and *z*, linked by a general bilinear relation

$$w = \frac{Az + B}{Cz + D}$$

We can write the relation as

$$w = \frac{\frac{A}{C}(Cz + D) + B - \frac{AD}{C}}{Cz + D} = \frac{A}{C} + \frac{B - \frac{AD}{C}}{Cz + D} \quad \Rightarrow \quad \frac{w - \frac{A}{C}}{B - \frac{AD}{C}} = \frac{1}{Cz + D}$$

If we consider two new variables: $W = \dfrac{w - \frac{A}{C}}{B - \frac{AD}{C}}$ $\qquad Z = Cz + D$

The relation becomes $\quad W = \dfrac{1}{Z}$

Therefore, a bilinear relation is just an hyperbolic relation after translation and zoom

Properties of a bilinear relation

If *z* portrays a circumference, also Z will portray a circumference (the transform is only a zoom followed by a translation), for example of radius *R* and center *S*. This condition is expressed by:

$$|Z-S| = R \Rightarrow (Z-S)(Z-S)^* = R^2 \Rightarrow ZZ^* - SZ^* - S^*Z + SS^* - R^2 = 0$$

By substituting $W = \frac{1}{Z}$ we get:

$$\frac{1}{WW^*} - \frac{S}{W^*} - \frac{S^*}{W} + SS^* - R^2 = 0 \Rightarrow WW^* - \frac{SW}{SS^* - R^2} - \frac{S^*W^*}{SS^* - R^2} + \frac{1}{SS^* - R^2} = 0$$

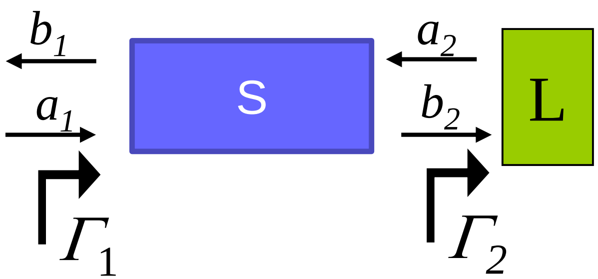
This relation still describes a circumference, with a new center S' and radius R'

$$S' = \frac{S^*}{SS^* - R^2} \qquad R'^2 = \frac{R^2}{(SS^* - R^2)^2}$$

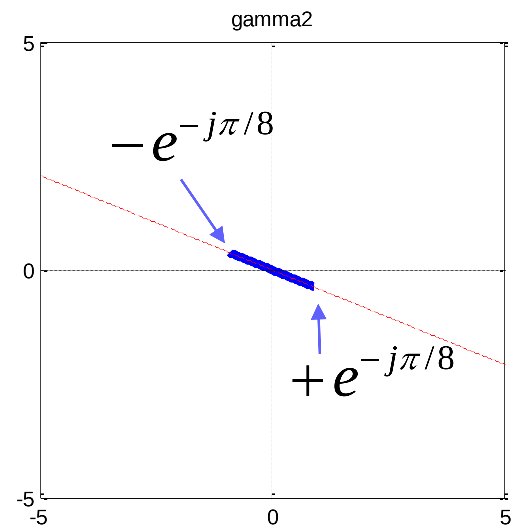
As conclusion, if *Z* follows a circumference, also *W* follows a circumference in the complex plane

Example of calculus with complex numbers: varying the Γ module

Consider a reflection coefficient Γ_2 varying from $-e^{-j\pi/8} \Rightarrow +e^{-j\pi/8}$



Γ_2



Lets calculate the evolution of Γ_1

After the network *S*

$$S = \begin{bmatrix} 0.6e^{i\pi/6} & 0.7e^{i\pi/3} \\ 0.7e^{i\pi/3} & 0.6e^{i\pi/6} \end{bmatrix}$$

The reflection coefficient becomes

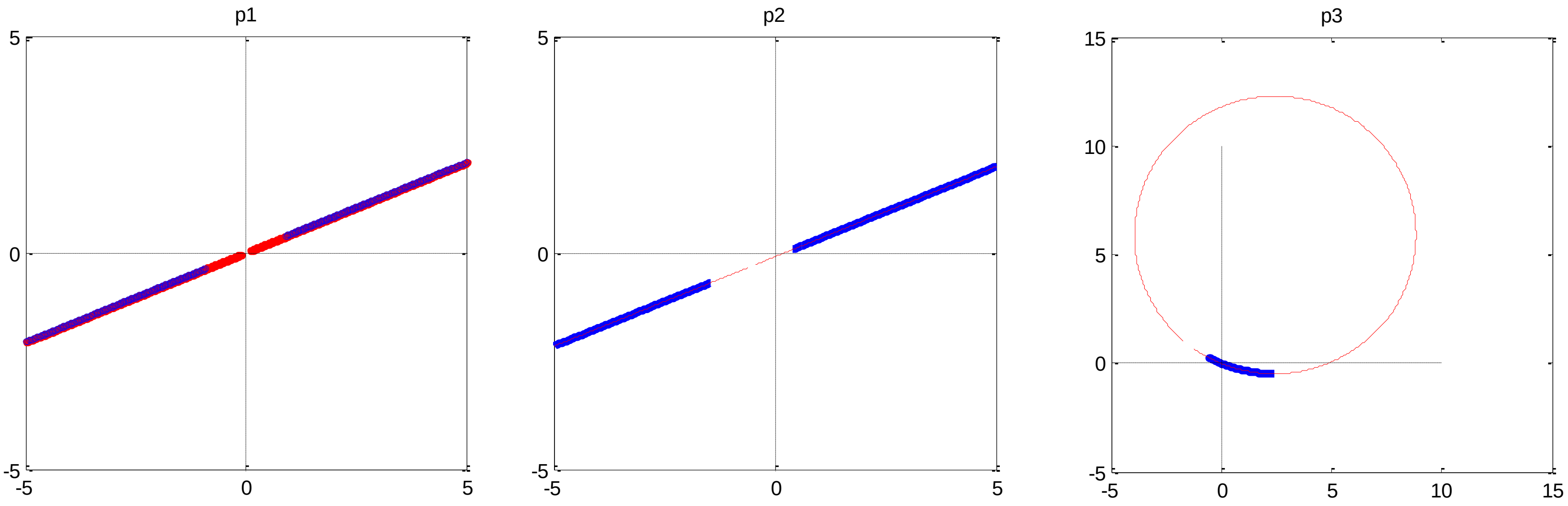
$$\Gamma_1 = S_{11} + \frac{S_{12}S_{21}}{\dfrac{1}{\Gamma_2} - S_{22}}$$

Example of calculus with complex numbers: varying the Γ module

$$p_1 = \frac{1}{\Gamma_2}$$

$$p_2 = p_1 - S_{22} = \frac{1}{\Gamma_2} - S_{22}$$

$$p_3 = \frac{1}{p_2} = \frac{1}{\frac{1}{\Gamma_2} - S_{22}}$$

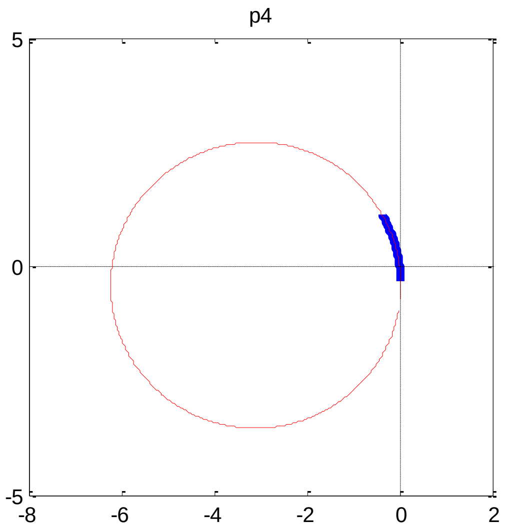


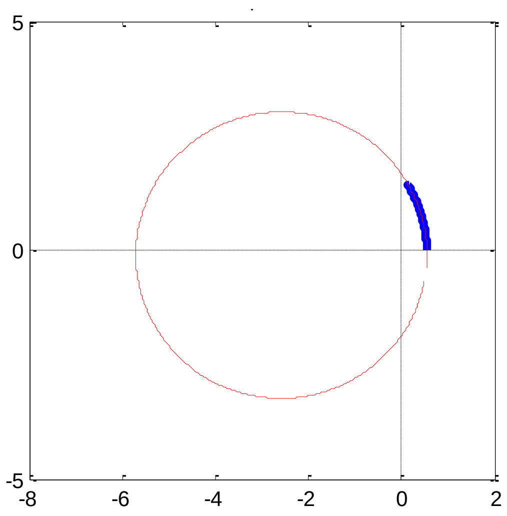
$$S = \begin{bmatrix} 0.6e^{i\pi/6} & 0.7e^{i\pi/3} \\ 0.7e^{i\pi/3} & 0.6e^{i\pi/6} \end{bmatrix}$$

Example of calculus with complex numbers: varying the Γ module

$$p_4 = S_{12}S_{21}p_3 = \frac{S_{12}S_{21}}{\frac{1}{\Gamma_2} - S_{22}}$$

$$\Gamma_1 = S_{11} + p_4 = S_{11} + \frac{S_{12}S_{21}}{\frac{1}{\Gamma_2} - S_{22}}$$

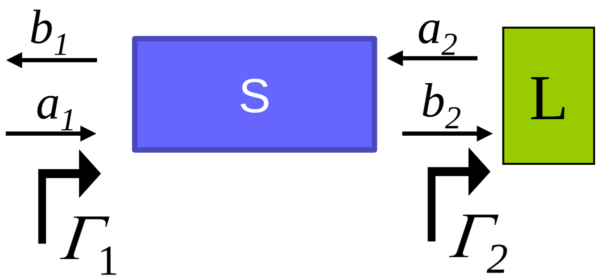


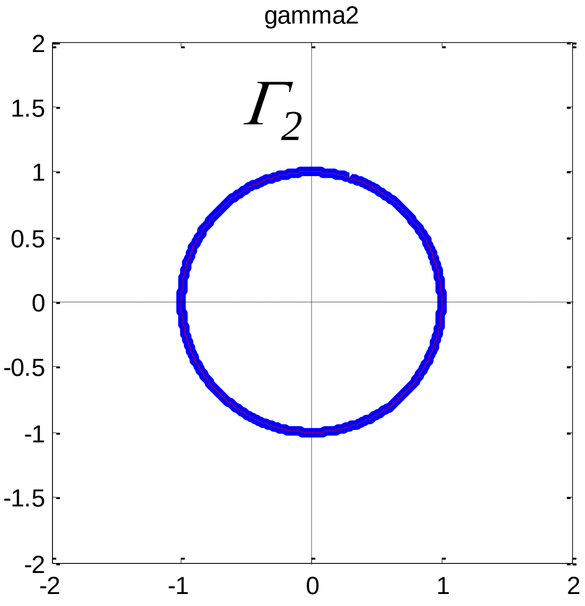


$$S = \begin{bmatrix} 0.6e^{i\pi/6} & 0.7e^{i\pi/3} \\ 0.7e^{i\pi/3} & 0.6e^{i\pi/6} \end{bmatrix}$$

Example of calculus with complex numbers: varying the Γ phase

Consider a reflection coefficient with $|\Gamma_2| = 1$ and phase varying from 0 to -2π





Lets calculate the evolution of Γ_1

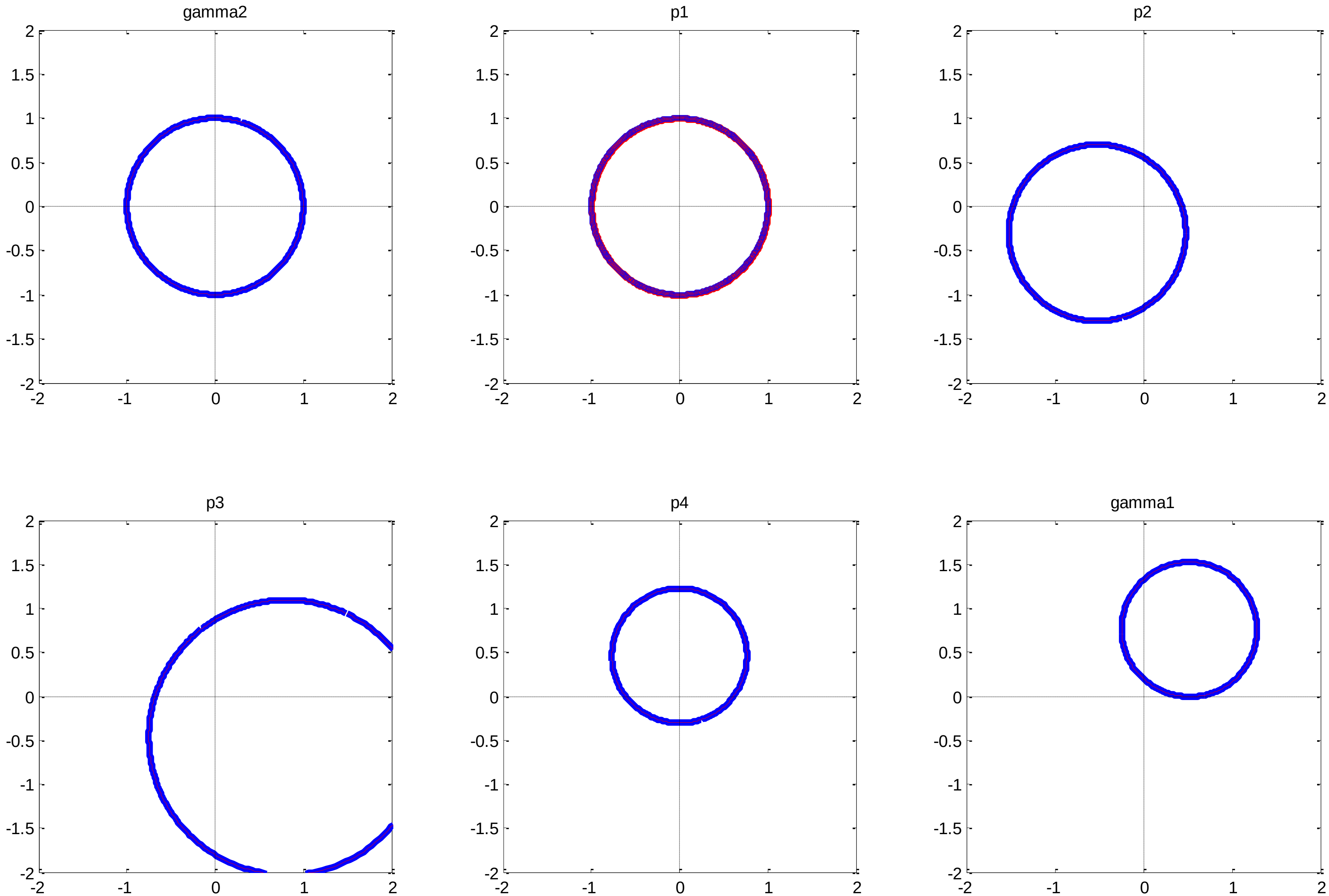
After the network *S*

$$S = \begin{bmatrix} 0.6e^{i\pi/6} & 0.7e^{i\pi/3} \\ 0.7e^{i\pi/3} & 0.6e^{i\pi/6} \end{bmatrix}$$

The reflection coefficient becomes

$$\Gamma_1 = S_{11} + \frac{S_{12}S_{21}}{\frac{1}{\Gamma_2} - S_{22}}$$

Example of calculus with complex numbers: varying the Γ phase



Exercise-1

- Calculate the scattering matrix of a transmission line perfectly matched, with length 21.5λ, and loss 0.1 dB.

The line is matched, therefore we have 0 on the main diagonal.

Certainly the line is reciprocal, then the matrix is symmetric.

The 0.1 dB loss in linear corresponds to $G = 10^{-(0.1/20)} = 0.988$

($20 \log_{10}(G) = 0.1$).

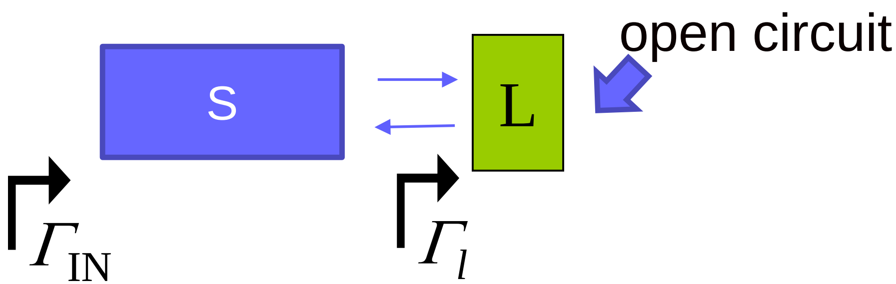
The phase of the transmission terms is given by

$$-\beta L = -\frac{2\pi}{\lambda} L = -\frac{2\pi}{\lambda} 21.5\lambda = -43\pi$$ Which corresponds to 180°

Resulting scattering matrix: $$S = \begin{bmatrix} 0 & -0.988 \\ -0.988 & 0 \end{bmatrix}$$

Exercise-2

- Calculate the reflection coefficient at the input of the line, when the output is left open (considering no radiation).



Let's apply the formula for the propagation of the reflection coefficient, remembering that the reflection coefficient of an open circuit is:

$$\Gamma_l = \frac{Z_{load} - Z_0}{Z_{load} + Z_0} = 1$$

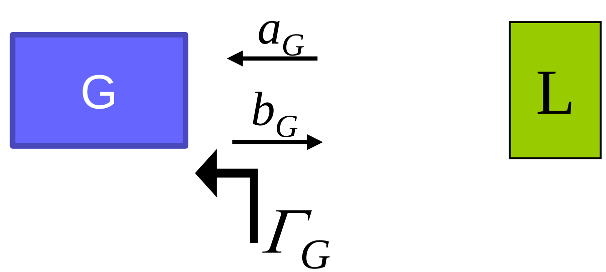
$$\Gamma_{IN} = S_{11} + \frac{S_{12}S_{21}\Gamma_l}{1 - S_{22}\Gamma_l} = 0 + \frac{0.988^2\,\Gamma_l}{1 - 0} = 0.976 \times \Gamma_l = 0.976$$

$$S = \begin{bmatrix} 0 & -0.988 \\ -0.988 & 0 \end{bmatrix}$$

Signal Generator

- The output signal from a generator can be written as a sum of a generated wave b_0 and a reflected wave (the generator could be not matched)

$$b_G = \Gamma_G a_G + b_0$$



- b_0 is the generated wave, obtained on a matched load

Matching Theorem: 3-port

It is not possible to realize a 3-port device without loss, reciprocal and completely matched.

Demonstration:

Consider a completely matched device $S_{11} = S_{22} = S_{33} = 0$

$$S = \begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix}$$

The condition of absence of loss is $\overline{[S]} \cdot [S]^* = [I]$

$$\begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{12} & 0 & S_{32} \\ S_{13} & S_{23} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & S_{12}^* & S_{13}^* \\ S_{21}^* & 0 & S_{23}^* \\ S_{31}^* & S_{22}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matching Theorem: 3-port

$$\begin{bmatrix} 0 & S_{21} & S_{31} \\ S_{12} & 0 & S_{32} \\ S_{13} & S_{23} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & S_{12}^* & S_{13}^* \\ S_{21}^* & 0 & S_{23}^* \\ S_{31}^* & S_{22}^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|S_{21}|^2 + |S_{31}|^2 = 1$$

$$S_{31} \cdot S_{32}^* = 0$$

$$S_{21} \cdot S_{23}^* = 0$$

$$|S_{12}|^2 + |S_{32}|^2 = 1$$

$$S_{12} \cdot S_{13}^* = 0$$

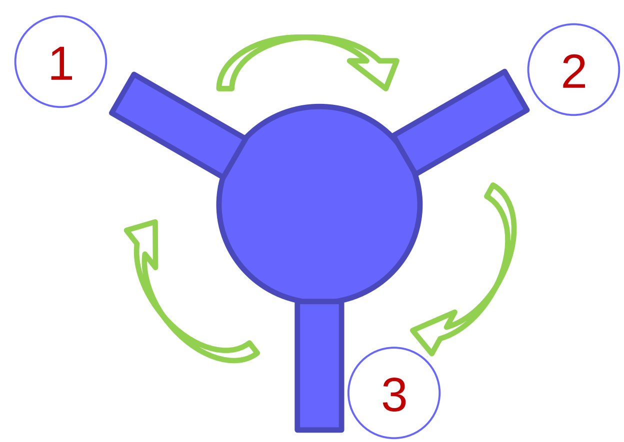
$$|S_{13}|^2 + |S_{23}|^2 = 1$$

There are only two possible solutions: or $S_{31} = 0$ or $S_{32} = 0$

$$S_{31} = 0 \rightarrow |S_{21}| = 1 \rightarrow S_{23} = 0 \rightarrow |S_{13}| = 1 \rightarrow S_{12} = 0 \rightarrow |S_{32}| = 1$$

$$S = \begin{bmatrix} 0 & 0 & e^{j\alpha} \\ e^{j\beta} & 0 & 0 \\ 0 & e^{j\gamma} & 0 \end{bmatrix}$$

Counterclockwise CIRCULATOR

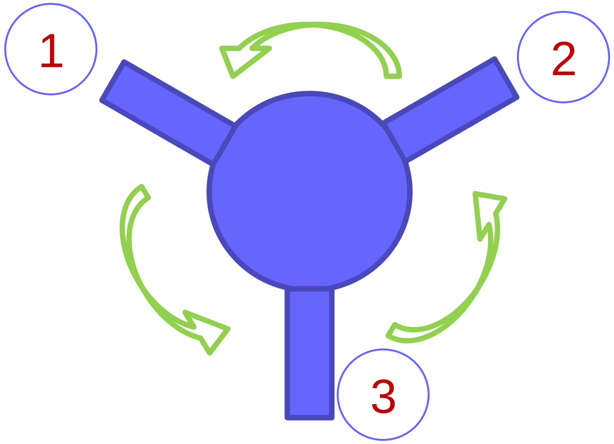


Matching Theorem: 3-port

Second possible solution: clockwise CIRCULATOR

$$S_{32}=0\rightarrow|S_{12}|=1\rightarrow S_{13}=0\rightarrow|S_{23}|=1\rightarrow S_{21}=0\rightarrow|S_{31}|=1$$

$$S=\begin{bmatrix}0 & e^{j\alpha} & 0\\ 0 & 0 & e^{j\beta}\\ e^{j\gamma} & 0 & 0\end{bmatrix}$$



$$|S_{21}|^2+|S_{31}|^2=1$$
$$S_{31}\cdot S_{32}^*=0$$
$$S_{21}\cdot S_{23}^*=0$$
$$|S_{12}|^2+|S_{32}|^2=1$$
$$S_{12}\cdot S_{13}^*=0$$
$$|S_{13}|^2+|S_{23}|^2=1$$

The circulator is lossless, perfectly matched, but not reciprocal

The input power at one port goes out from the next port

Matching Theorem: 4-port

Let consider a 4-port device reciprocal and without loss. If two port are matched and not coupled (there is no power transfer between them), the other two port are also matched and not coupled.

Demonstration:

Consider a reciprocal device (symmetric matrix), with two port matched and not coupled

$$S = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & 0 & 0 \\ S_{14} & S_{24} & 0 & 0 \end{bmatrix}$$

The condition of absence of loss is $\overline{[S]} \cdot [S]^* = [I]$

We use only the four equations given by the main diagonal of the identity matrix

Matching Theorem: 4-port

We use only the four equations given by the main diagonal of the identity matrix

$$\overline{[S]}\cdot[S]^* = [I] \quad \Rightarrow \quad \begin{aligned} |S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 &= 1 \\ |S_{12}|^2 + |S_{22}|^2 + |S_{23}|^2 + |S_{24}|^2 &= 1 \\ |S_{13}|^2 + |S_{23}|^2 &= 1 \\ |S_{14}|^2 + |S_{24}|^2 &= 1 \end{aligned}$$

By adding the first two equation and the second two equation we get:

$$|S_{11}|^2 + 2|S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 + |S_{22}|^2 + |S_{23}|^2 + |S_{24}|^2 = 2$$

$$|S_{13}|^2 + |S_{23}|^2 + |S_{14}|^2 + |S_{24}|^2 = 2$$

By subtracting the two equation we get

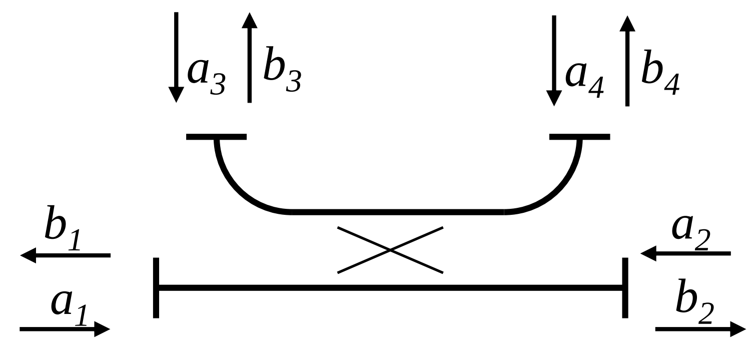
$$|S_{11}|^2 + 2|S_{12}|^2 + |S_{22}|^2 = 0 \quad \Rightarrow \quad |S_{11}|^2 = |S_{12}|^2 = |S_{22}|^2 = 0$$

Demonstrated!

IDEAL DIRECTIONAL COUPLER

The device that satisfies the 4-port matching theorem is the **directional coupler**. The ideal directional coupler is without loss, reciprocal, and two couple of port are not coupled. Considering a symmetric structure, the scattering matrix can be written as

$$S = \begin{bmatrix} 0 & \sqrt{1-|k^2|}e^{j\varphi} & 0 & k \\ \sqrt{1-|k^2|}e^{j\varphi} & 0 & k & 0 \\ 0 & k & 0 & \sqrt{1-|k^2|}e^{j\varphi} \\ k & 0 & \sqrt{1-|k^2|}e^{j\varphi} & 0 \end{bmatrix}$$



The coupling coefficient is $S_{41} = S_{14} = k$

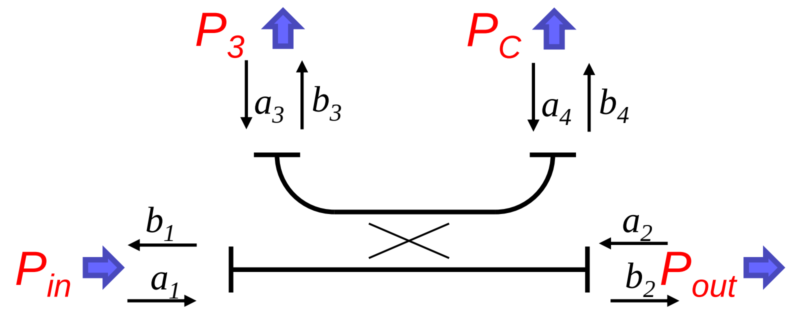
The transmission coefficient is $S_{21} = S_{12} = \sqrt{1-|k^2|}e^{j\varphi}$

REAL DIRECTIONAL COUPLER

The coupling factor is

$$K = 10 \log_{10}(P_{in}/ P_C)= 20 \log_{10} | 1 / S_{41} |$$

It is typically the first feature of the coupler



The transmission loss is

$$L = 10 \log_{10} (P_{in}/ P_{out}) = 20 \log_{10} | 1 / S_{21} |$$

It is typically the first feature of the coupler

The isolation is

$$I = 10 \log_{10} (P_{in}/ P_3)= 20 \log_{10} | 1 / S_{31} |$$

It should be infinite in an ideal coupler

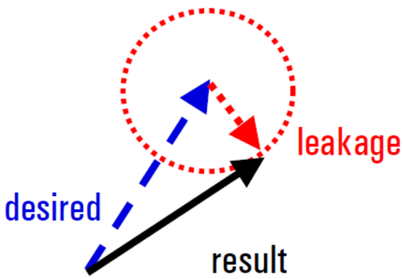
The directivity is

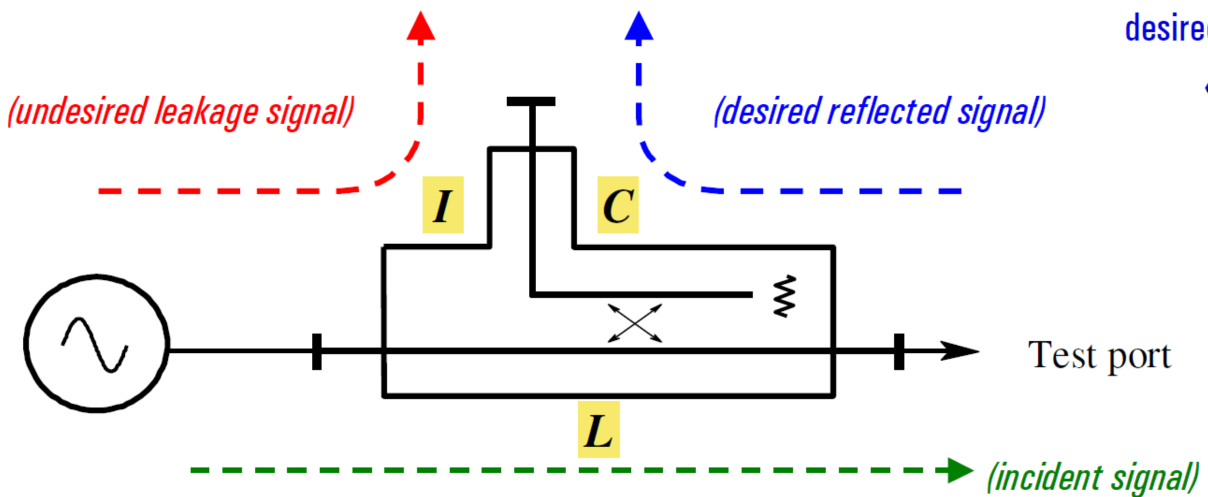
$$D(\text{dB}) = I(\text{dB}) - K(\text{dB}) - L(\text{dB}) = 20 \log_{10} | S_{21}S_{32}/S_{31} |$$

It is the measurement of the ratio between the signals at port 3: the one reflected by a load with $|\Gamma|=1$ placed at port 2 (desired), and the one coupled between port 3 and 1 (undesired).

DIRECTIONAL COUPLER DIRECTIVITY

Directivity is a measure of how well a coupler can separate signals moving in opposite directions





Directivity = Isolation (I) - Fwd Coupling (C) - Main Arm Loss (L)